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THE ROLE OF MONEY IN SUPPORTING THE PARETO OPTIMALITY OF COMPETITIVE  
EQUILIBRIUM IN CONSUMPTION-LOAN MODELS\*

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## I. INTRODUCTION AND SUMMARY

Perhaps the single most enduring theme in economics is that of the social desirability of the competitive mechanism. In its modern form, this theme occurs as the two basic theorems of welfare economics (see, in particular, Arrow [1]). Our central concern in this paper is with the validity of the first of these two theorems -- that every competitive equilibrium yields a Pareto optimal allocation -- in idealized yet plausible models of intertemporal allocation in a market economy.

What is especially striking about the posture of the "invisible hand" is its apparent widespread reach; all that seems to be required in the well-known standard argument is that there be neither real externalities (in consumption and production) nor local satiation (in consumption). This is very misleading -- quite aside from any questions of the existence of competitive equilibrium. One of the most important features of the maintained assumptions in that argument is that they implicitly impose some element of boundedness in order to offset the intrinsic one-directional, open-ended nature of time. In the commonly accepted paradigm, this element is simply imposed by postulating a bounded horizon (see, for example, Arrow-Hahn [2]). In another frequently recurring extension, it is effectively imposed by postulating the alternative -- and equally implausible -- assumption that there are essentially a finite number of infinitely-lived agents (see, for example, Debreu [5]). In any case, it is not

necessarily true that the "invisible hand" stretches over economies whose evolution extends towards an unbounded horizon -- even in the most favorable of circumstances otherwise.

This significant exception was first recognized, at least implicitly, in Malinvaud's classic paper on capital theory [8]. However, it only received its first explicit elaboration in Samuelson's equally celebrated seminal contribution [10]. Samuelson's discussion is the starting point for our own.

Samuelson showed that -- in a simple model of a market economy characterized by an unbounded horizon, short-lived, overlapping but essentially identical households and a single, perishable physical commodity -- without some extra-market institution, there may be no competitive equilibrium which is Pareto optimal. He also showed that one natural extra-market institution which may set matters aright is fiat money initially owned (i.e., cleverly invented) by the first generation of households, and subsequently purchased with physical commodities (i.e., expressly valued) by each succeeding generation -- provided only that money trades for commodities at a sufficiently high (present value) price. A number of others have since refined and extended Samuelson's central argument (see, for example, Cass-Kaari [3], Diamond [6], Gale [7], Shell [11] and Starrett [14]).

Even on his own grounds Samuelson dealt with a special case (boundary endowments) in a special way (stationary allocations). Expanding his analysis -- much in the manner of Gale, but in different spirit -- it is easy to show that in Samuelson's basic model (with two-period lifetimes) there is the following dichotomy: Without money,

the competitive equilibrium is the initial endowment allocation itself, which may be Pareto optimal or not. Furthermore, upon the introduction of money, in the former case nothing is altered. The price of money must be zero and the allocation is again the initial endowments (in this instance, Pareto optimal). In the latter case, however, there now exists a continuum of potential competitive equilibria. Considering just the least complicated possibility, these range from the original autarkic equilibrium -- where the price of money turns out to be zero and the allocation is once again the initial endowments (in this instance, not Pareto optimal) -- to the "fully" monetized equilibrium -- where the price of money is at a maximum and the allocation turns out to be both stationary and Pareto optimal. In the intermediate range -- that of the "partly" monetized equilibria -- the price of money is positive, but the allocations are neither stationary nor Pareto optimal. In fact, in the "partly" monetized equilibria, the allocation must be asymptotically the same as in the original autarkic equilibrium.

Thus, a complete analysis of Samuelson's basic model (now including consideration of more complicated possibilities for nonstationary equilibria) leads to several very strong conclusions. These can be summarized, in somewhat general fashion, by the following propositions: Consider the range of potential competitive equilibria with money, and call an equilibrium barter if the price of money is zero, monetary if it's positive. Then we have

Existence Proposition: There is a monetary equilibrium if and only if there is no barter equilibrium which is Pareto optimal.

Optimality Proposition: If there is a monetary equilibrium, then there is also a monetary equilibrium which is Pareto optimal.

Note, in particular, that both propositions together imply that there is always some competitive equilibrium (barter or monetary) which is Pareto optimal.

The central issue we address in this paper is how robust these propositions (and their related implications) are to various generalizations of Samuelson's basic model, and especially to relaxing the assumption that all households are essentially identical (i.e., have the same tastes for and endowments of physical commodities but for date of birth).

It would hardly be surprising if such relaxation required some qualification or modification of the propositions. It is quite surprising, however, that with the introduction of what amounts to fairly routine variety in tastes and endowments -- judged by that typically encountered in general equilibrium theory -- neither proposition survives in any recognizable form. Specifically, we show that the consumption-loan model with heterogeneous households (and, upon occasion, other extensions) yields the following sorts of counterexamples:

Coexistence Example: There are both barter and monetary equilibria which are Pareto optimal.

Nonoptimality Example: There are both barter and monetary equilibria, but none which is Pareto optimal.

While there is some degree of speciality to our examples, especially those exhibiting "nonoptimality", this seems dictated more by our maintained simplifying assumptions -- that there is effectively just a single common good (though perhaps more than one physical commodity) in each period, and that each household survives at most two periods -- than by any inherent feature of the issues involved.

The significance of the coexistence examples may not be immediately apparent. But, in fact they do carry an important message. On the one hand, these examples clearly demonstrate that there is no observable criterion for determining whether the existence of fiat money as a store of value is necessary to support the Pareto optimality of competitive equilibrium. On the other hand, much more critically, they graphically illustrate one basic difficulty encountered in assigning a normative role to fiat money -- the wide extent of nonuniqueness of monetary equilibrium: Even when the presence of money (trading for commodities at some suitable positive price) can surely guarantee the Pareto optimality of competitive equilibrium, the competitive mechanism by itself offers no assurance whatsoever that it will. Indeed, our coexistence examples strongly suggest that -- in general -- something like continuous monitoring of the price level may be an indispensable component of an otherwise neutral monetary (or more accurately, fiscal) policy.

It is the nonexistence examples, however, which convey the central message we have to communicate. These examples dramatically highlight a second, even more fundamental difficulty with relying on the mere creation of fiat money to conjure up an effective appeal to

the second basic theorem of welfare economics -- the limited scope of once-and-for-all augmentation of initial wealth: Just the presence of money (trading for commodities at any conceivable positive price) may possibly not guarantee the Pareto optimality of competitive equilibrium. Indeed, our nonoptimality examples strongly suggest that -- in general -- something like continuous redistribution by means of creation (or destruction) of fiat money may be an indispensable lubricant for the efficient operation of an evolving market economy.

The plan of the paper is as follows: A partially generalized consumption-loan model is described in Section II. Section III contains a review of the leading special case, Samuelson's basic model. The core examples, exhibiting the coexistence of both barter and monetary equilibria which are Pareto optimal, and the nonexistence of any competitive equilibrium which is Pareto optimal, are presented in Sections IV and V, respectively. Finally, the Appendix contains some technical results we require involving the construction of offer curves (or equivalently, indifference maps) exhibiting various special properties.

## II. THE BASIC MODEL

The economy begins operation in period 1, and continues over periods extending indefinitely into the future  $t = 1, 2, \dots$ . In each period there are two commodities, one a perishable physical good (whose various quantities are subscripted by the period in which it is available), the other an imperishable fiat money. Households or consumers (whose various attributes are superscripted by the order

in which they are born, say,  $h = 0, 1, \dots$ ) are either present at the inception of the economy -- in which case they live out the balance of their lives during period 1 -- or born at the beginning of each period  $t \geq 1$  -- in which case they live out the whole of their lives during that and the succeeding period. Thus, in each period there are always just two age groups of consumers, an older generation born in the preceding period, say,  $G_{t-1}$ , and a younger generation born in the current period  $G_t$ . For the most part we will only be concerned with one or the other of the two simplest conceivable demographic patterns, namely, that either  $G_t = \{t\}$  for  $t \geq 0$  -- each generation consists of a single consumer -- or  $G_0 = \{0\}$  and  $G_t = \{2t-1, 2t\}$  for  $t \geq 1$  -- the oldest generation consists of a single consumer, and each succeeding generation of two consumers.<sup>1</sup>

Each consumer in each generation  $h \in G_t$ ,  $t \geq 0$  (potentially) derives satisfaction or utility  $U^h$  from consuming goods during his lifetime  $c^h = c_1^h$  for  $h \in G_0$  and  $c^h = (c_t^h, c_{t+1}^h)$  for  $h \in G_t$ ,  $t \geq 1$ . This fundamental economic hypothesis is represented by a utility function  $U^h = U^h(c^h)$  for  $c^h \geq 0$ , herein typically assumed to be at least continuous, quasi-concave (i.e., to exhibit a diminishing marginal rate of substitution) and to have no local maxima (i.e., to exhibit local nonsatiation).<sup>2</sup> Each consumer also has given endowments of the goods available during his lifetime  $y^h = y_1^h > 0$  for  $h \in G_0$  and  $y^h = (y_t^h, y_{t+1}^h) \geq 0$  for  $h \in G_t$ ,  $t \geq 1$ , while each consumer in the oldest generation has a given endowment of money  $m^h > 0$  for  $h \in G_0$ .

We assume that the total amount of money in the economy consists of one unit  $\sum_{h \in G_0} m^h = 1$  (so that if  $G_0 = \{0\}$ , then  $m^0 = 1$ ), which amounts

to defining the monetary unit. Finally, each consumer can buy and sell (within physical and temporal capabilities) either goods or money on both a spot and a one-period futures market at perfectly foreseen (present value) prices, denoted  $p_t$  and  $p_m$ , respectively.<sup>3</sup>

Given these opportunities, he chooses his lifetime consumption profile rationally, that is, as some optimal solution to the budget constrained utility maximization problem

$$(1) \begin{cases} \text{maximize} & U^h(c_1^h) \\ \text{subject to} & c_1^h \leq y_1^h + p_m m^h \\ & c_1^h \geq 0 \end{cases} \quad \text{for } h \in G_0$$

and

$$(2) \begin{cases} \text{maximize} & U^h(c^h) \\ \text{subject to} & p_t c_t^h + p_{t+1} c_{t+1}^h \leq p_t y_t^h + p_{t+1} y_{t+1}^h \\ & c^h \geq 0 \end{cases} \quad \text{for } h \in G_t, t \geq 1.$$

Aggregate consistency in these choices completes the model. A competitive equilibrium is a set of positive goods prices, together with a nonnegative price of money, and optimal lifetime consumption profiles satisfying market clearing

$$(3) \quad \sum_{h \in G_{t-1}} c_t^h = \sum_{h \in G_{t-1}} y_t^h \quad \text{or} \quad \sum_{h \in G_{t-1}} (c_t^h - y_t^h) = \sum_{h \in G_t} -(c_t^h - y_t^h) \quad \text{for } t \geq 1.$$

As suggested in the introduction, an important distinction will be that between a barter equilibrium -- a competitive equilibrium in which  $p_m = 0$  -- and a monetary equilibrium -- a competitive equilibrium in which  $p_m > 0$ . Also, we will occasionally refer to the set of lifetime consumption profiles corresponding to some competitive equilibrium as a competitive allocation. In contrast, a feasible allocation (explicitly taking the notion of perishable physical goods to entail free disposal) is simply a set of nonnegative lifetime consumption profiles satisfying materials balance

$$(4) \quad \sum_{h \in G_{t-1}} c_t^h \leq \sum_{h \in G_{t-1}} y_t^h \quad \text{for } t \geq 1.$$

To complete the list of standard definitions in this context, we define a particular feasible allocation (for instance, some competitive allocation), say,  $\{c^{h'}\}$ , to be Pareto optimal if there is no other feasible allocation  $\{c^{h''}\}$  such that

$$(5) \quad \begin{cases} U^h(c^{h''}) \geq U^h(c^{h'}) \text{ for all } h \geq 0 \\ \text{and} \\ U^h(c^{h''}) > U^h(c^{h'}) \text{ for some } h \geq 0. \end{cases}$$

The foregoing description of an economy generally differs from the canonical consumption-loan model in only one significant

respect, namely, in admitting the possibility of heterogeneity in tastes for and endowments of goods both within and across generations. While it is this degree of freedom which plays a dominant role in our analysis, there are several additional, minor variations which we will also call upon for support: (i) Availability of a second perishable physical commodity: Formally, such a commodity can be accounted for merely by reinterpreting  $c, y$  and  $p$  as 2-vectors. However, for our particular purposes it will be less confusing to introduce a more clearly distinct notation for the quantity, endowment and (present value, i.e., with the first commodity in period 1 as numeraire) price of this second commodity,  $x, w$  and  $q$ , respectively. (ii) Shorter (or longer) lifetimes: For this extension we will simply adopt the formality of reinterpreting  $c^h$  and  $y^h$  -- or  $(c^h, x^h)$  and  $(y^h, w^h)$  -- and their aggregate counterparts as needs be. (iii) Money endowments to consumers in other than the oldest generation: This extension is easily accomplished by specifying that such endowments occur in the second period of life  $m^h = m_{t+1}^h$  for  $h \in G_t$ ,  $t \geq 1$ , and including their present value on the righthand side of the budget constraint in (2). (In particular instances, some of these endowments may be negative, or may correspond to "taxes" rather than "subsidies".)

The virtue -- and, as with moral rectitude, limitation -- of our basic model is that, because there is only a single good available, and a single period overlap, the set of potential competitive equilibria can be succinctly characterized. In particular, define (presupposing (1), (2) and (3))

$z_t$  = excess demand by generation  $t-1$  for the good in their second period of life

$$= \sum_{h \in G_{t-1}} (c_t^h - y_t^h)$$

$$= \sum_{h \in G_t} -(c_t^h - y_t^h)$$

= excess supply by generation  $t$  of the good in their first period of life

and

$$g_t = \{(z_t, z_{t+1}) : (z_t, z_{t+1}) = \sum_{h \in G_t} (-(c_t^h - y_t^h), (c_{t+1}^h - y_{t+1}^h))\}$$

such that  $(c_t^h, c_{t+1}^h)$  is an optimal solution

to (2) for some  $(p_t, p_{t+1}) > 0$

= reflected generational offer curve of generation  $t$

for  $t \geq 1$ . Now note that (i)  $z_1 = \sum_{h \in G_0} (c_1^h - y_1^h) = \sum_{h \in G_0} p_m^h = p_m$ ,

while (ii) by suitable choice of the units for measuring the good

in each period,  $\sum_{h \in G_t} y_t^h = y > 0$  for  $t \geq 1$ .<sup>4</sup> Then, it is easily seen

that the set of potential competitive equilibria is essentially

equivalent to the set of solutions to the fundamental dynamical system

$$(6) \quad \begin{cases} z_1 \geq 0 \\ (z_t, z_{t+1}) \in g_t \text{ and } z_t \leq y \text{ for } t \geq 1. \end{cases}^5$$

In other words, given the basic data describing population structure  $G_t$  for  $t \geq 0$ , and individual tastes and endowments  $U^h$  and  $y^h$  for  $h \in G_t$ ,  $t \geq 0$ , the potential evolution of the economy is completely captured, in terms of the "reduced" data describing reflected generational offer curves  $g_t$  for  $t \geq 1$  and aggregate first period endowment  $y$ , by means of the system (6). For future reference, it will be useful to bear in mind that, except possibly for boundary endowments,  $0 \in g_t$  for  $t \geq 1$  (so that typically  $z_t = 0$  for  $t \geq 0$  is a solution to (6) -- which means, of course, that there is some barter equilibrium), while if  $(z_t, z_{t+1}) \in g_t$ , then

$$z_{t+1} \begin{cases} > \\ = \end{cases} 0 \text{ according as } z_t \begin{cases} > \\ = \end{cases} 0$$

(so that every solution to (6) is nonnegative -- which is, of course, our prime motivation for employing the reflected offer curve rather than the standard offer curve itself). Moreover, perhaps most importantly, points on the reflected generational offer curve which also represent competitive allocations must satisfy the equation

$$z_{t+1}/z_t = p_t/p_{t+1} = 1 + r_t = \text{one plus the real rate of return from period } t \text{ to period } t+1.^6$$

The core of our analysis involves focusing on a series of special cases, that is, detailed specifications of  $G_t$ ,  $U^h$  and  $y^h$  and hence  $g_t$  and  $y$ , and answering the following sorts of fundamental questions:

1. What are the solutions to (6), and thus (1)-(3)? In particular, (knowing that there is a barter equilibrium) is there a monetary equilibrium? In fact (assuming replication in the basic data), a stationary monetary equilibrium?

and

2. What properties do these solutions exhibit? In particular, is some barter equilibrium Pareto optimal? If not, is some monetary equilibrium?

Partly as an exercise to gain familiarity with technique, but mostly as a review to provide foundation for comparison, we turn first to utilizing (6) to analyze the competitive equilibria in Samuelson's basic model.

### III. SAMUELSON'S BASIC MODEL

Samuelson's central story (and more) can be fully elaborated in terms of the leading special case of our model, where (i) each generation consists of just a single consumer, so that  $G_t = \{t\}$  for  $t \geq 0$ , and (ii) each consumer but the oldest has the same utility function, so that  $U^t(c^t) = U(c^t)$  for  $t \geq 1$  -- where  $U$  is now assumed to be differentiable, strictly quasi-concave and strictly increasing -- as well as the same positive endowments, so that  $y^t = (y_1, y_2) > 0$  for  $t \geq 1$ .<sup>7</sup> The critical feature of this case is that, besides being stationary, the reflected generational offer curve derives from the rational behavior of the representative consumer, so that

$$g_t = g = \{(z_1, z_2) : (-z_1, z_2) \geq -(y_1, y_2), \frac{\partial U(-z_1+y_1, z_2+y_2)}{\partial c_1} / \frac{\partial U(-z_1+y_1, z_2+y_2)}{\partial c_2} = p_1/p_2\},$$

$$\text{and } -p_1 z_1 + p_2 z_2 = 0 \text{ for some } (p_1, p_2) > 0\}$$

for  $t \geq 1$ , as illustrated in Figures 1 and 2.

From careful study of these figures, it becomes apparent that this particular feature has two important consequences for the structure of  $g$ , the solutions to (6), and thus the nature of competitive equilibrium: First,  $g$  intersects the origin just once. In other words, there is a unique (stationary) barter equilibrium, supported by prices  $p_1 = 1$ ,  $p_t/p_{t+1} = 1 + r_t = 1 + r$  for  $t \geq 1$  and  $p_m = 0$ , and yielding the autarkic allocation  $c_1^0 = y_1^0$  and  $(c_t^t, c_{t+1}^t) = (y_1, y_2)$  for  $t \geq 1$ . (Here, as in what follows, we utilize notation which is either formally defined in the text or informally defined in the various accompanying figures.) Second, in addition,  $g$  intersects the 45° line in the positive quadrant, just once, if and only if it has slope less than one at the origin. In other words, there will also be a unique stationary monetary equilibrium, supported by prices  $p_1 = 1$ ,  $p_t/p_{t+1} = 1 + r_t = 1$  for  $t \geq 0$  and  $p_m = p_m^*$ , and yielding the trading allocation  $c_1^0 = y_1^0 + (y_1 - c_1^*)$  and  $(c_t^t, c_{t+1}^t) = (c_1^*, c_2^*)$  for  $t \geq 1$  -- where  $p_m^* = y_1 - c_1^* > 0$  or  $c_1^* < y_1$  and  $c_2^* > y_2$  -- if and only if  $1 + r < 1$ . In short, the dichotomy emphasized earlier in Section I ultimately depends simply on the magnitude of the representative consumer's marginal rate of substitution at his endowments



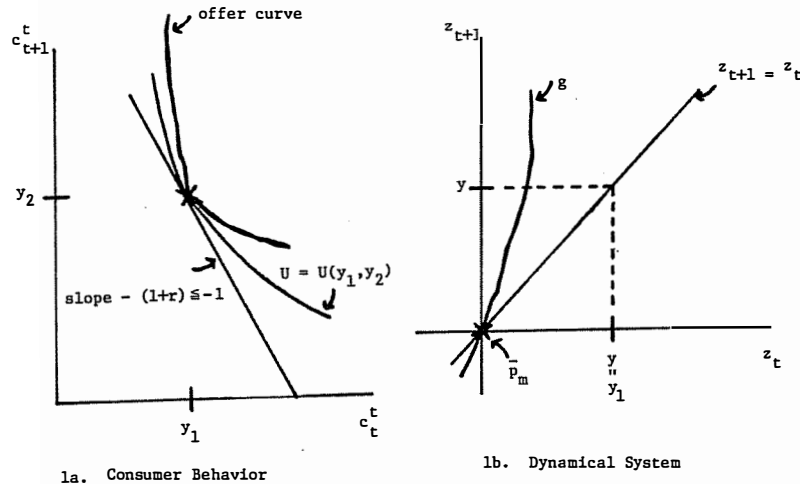


Figure 1. Potential Competitive Equilibria in Samuelson's Basic Model:  
Only a Single Barter Equilibrium  $\left( \frac{\partial U(y_1, y_2)}{\partial c_1} / \frac{\partial U(y_1, y_2)}{\partial c_2} = 1 + r \geq 1 \right)$

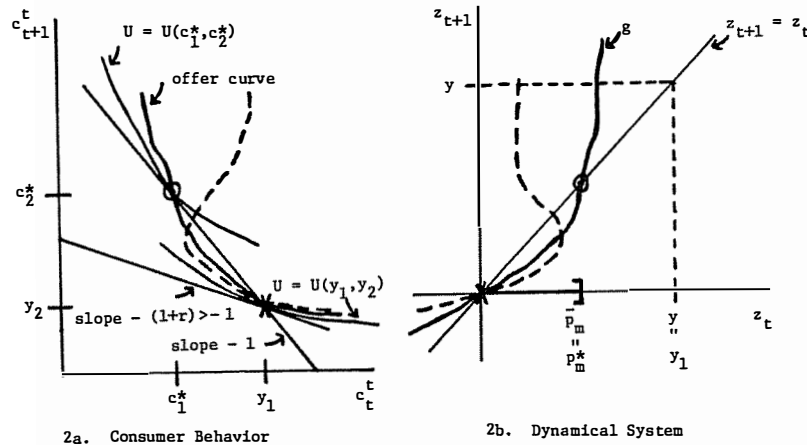
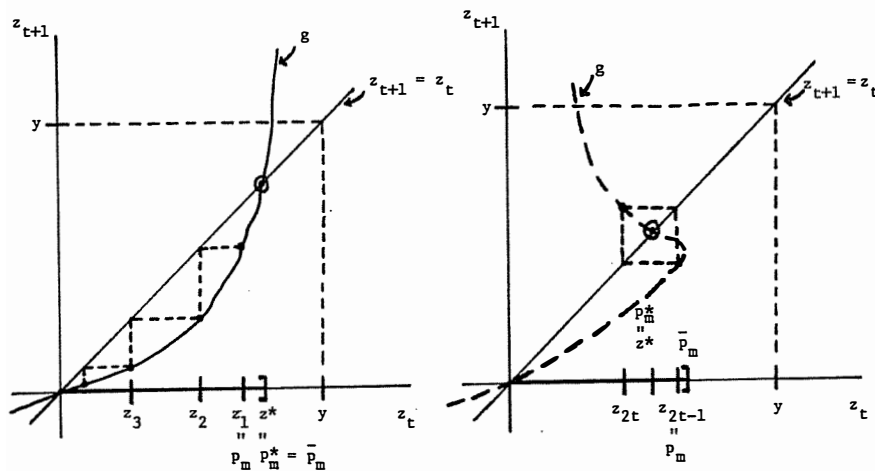


Figure 2. Potential Competitive Equilibria in Samuelson's Basic Model:  
Also a Continuum of Monetary Equilibria  $\left( \frac{\partial U(y_1, y_2)}{\partial c_1} / \frac{\partial U(y_1, y_2)}{\partial c_2} = 1 + r < 1 \right)$

$$\frac{\partial U(y_1, y_2)}{\partial c_1} / \frac{\partial U(y_1, y_2)}{\partial c_2} = 1 + r.$$

Continuing with an explicit description of that dichotomy -- and with heavy reliance on graphical intuition and argument -- we notice next that, in fact, there will be a whole range of monetary equilibria, corresponding to prices of money satisfying  $0 < \bar{p}_m \leq \bar{p}_m^*$ , if and only if there is a stationary monetary equilibrium, that is, once again,  $1 + r < 1$ .<sup>8</sup> This situation is suggestively exemplified in Figure 3, which contrasts the two possible cases: On the one hand, in the "normal" case (pictured in Figure 3a), we have  $\bar{p}_m = p_m^*$ , so that (i) if  $p_m = \bar{p}_m$ , then competitive equilibrium is necessarily the stationary monetary one, while (ii) if  $p_m < \bar{p}_m$ , then competitive allocation necessarily converges monotonically to the representative consumer's endowments. On the other hand, in the "abnormal" case (pictured in Figure 3b), we have  $\bar{p}_m > p_m^*$ , so that evidently matters are not nearly so transparent. Indeed, in this case, there generally need not be just a single competitive equilibrium corresponding to each (sufficiently large) price of money, nor even any recognizable pattern to the asymptotic behavior of competitive allocation -- even though we have chosen to depict a competitive equilibrium which replicates cyclically every two periods.<sup>9</sup>

These various cases (i.e.,  $1 + r \geq 1$  or  $1 + r < 1$ ) and "normality" or "abnormality") have their counterparts in terms of welfare implications. In particular, using the fact that  $U$  is assumed strictly quasi-concave -- so that transfers from consumer  $t + 1$  to consumer  $t - 1$  via consumer  $t$ , transfers which also (at least) maintain



3a. The "Normal" Case

3b. The "Abnormal" Case

Figure 3. Examples of both Stationary (●) and Nonstationary (•••)  
Monetary Equilibria in Samuelson's Basic Model

the intermediary consumer's welfare, necessarily involve increasingly unfavorable real terms of trade between good  $t$  and good  $t+1$  -- it can be easily demonstrated that if  $1+r \geq 1$ , then the barter equilibrium is Pareto optimal, while if  $1+r < 1$ , then the stationary monetary equilibrium is Pareto optimal.<sup>10</sup> In contrast, but again explicitly using the fact that  $U$  is assumed strictly quasiconcave -- so that  $U(c_1, c_2) < U(c_1^*, c_2^*)$  for every  $c_1 + c_2 \leq y_1 + y_2$  such that  $(c_1, c_2) \neq (c_1^*, c_2^*)$  -- if  $1+r < 1$ , then in the "normal" case the allocation corresponding to each competitive equilibrium (barter or monetary) except the stationary monetary one is strictly dominated by that corresponding to the latter. In other words, in this case, neither the barter equilibrium nor any nonstationary monetary equilibrium is Pareto optimal.

For the same basic reason, in the "abnormal" case, neither the barter equilibrium nor any nonstationary monetary equilibrium yielding the same (endowments) allocation asymptotically -- that is, such that  $\lim_{t \rightarrow \infty} z_t = 0$  -- is Pareto optimal. But in this case, the general welfare status of the nonstationary monetary equilibria is mixed, since it can be proven that every competitive equilibrium such that  $\limsup_{t \rightarrow \infty} z_t > 0$  is in fact Pareto optimal -- including, for instance, the specific periodic equilibrium depicted in Figure 3b. (We omit the proof of this assertion, which we also refer to in the following section; the argument is quite straightforward but rather tedious.)

Given our present purposes, the foregoing descriptive analysis of Samuelson's basic model has already been aptly summarized

by the two Propositions stated in Section I. We now proceed to the heart of the paper, to the demonstration that these properties are actually very model-bound. Before doing so, however, we might well underline that in each of the following arguments we will utilize essentially the same graphical heuristic utilized here (though bolstered now and then by analytical means). We take this rehearsal as providing license to be somewhat terse in presentation, if not also in interpretation.

#### IV. COEXISTENCE OF BOTH BARTER AND MONETARY EQUILIBRIA WHICH ARE PARETO OPTIMAL

The model underlying the two examples presented in this section differs in only one essential respect from Samuelson's basic model, to wit, in that there are two distinct types of consumers in every generation but the oldest. More specifically, suppose now that  $G_0 = \{0\}$  and  $G_t = \{2t-1, 2t\}$  for  $t \geq 1$ , and that the odd-numbered consumers are of  $\alpha$ -type, so that  $U^h(c^h) = U^\alpha(c^h)$  and  $y^h = y^\alpha > 0$  for  $h = 2t-1$ ,  $t \geq 1$ , the even-numbered of  $\beta$ -type, so that  $U^h(c^h) = U^\beta(c^h)$  and  $y^h = y^\beta > 0$  for  $h = 2t$ ,  $t \geq 1$  -- where generally  $U^\alpha \neq U^\beta$  and  $y^\alpha \neq y^\beta$ . Even this seemingly minor modification, by permitting greater freedom in specifying the reflected generational offer curves (which are, as before, stationary over time  $g_t = g$  for  $t \geq 1$ ), entails fundamentally contrary consequences for the conclusions outlined in the preceding section. The most important of these, and the one we detail here, is illustrated by the two examples shown in Figures 4 and 5.

The key feature of these counterexamples is that in the first (resp., second) (i) when the real rate of return is zero,  $p_1/p_2 = 1$ ,

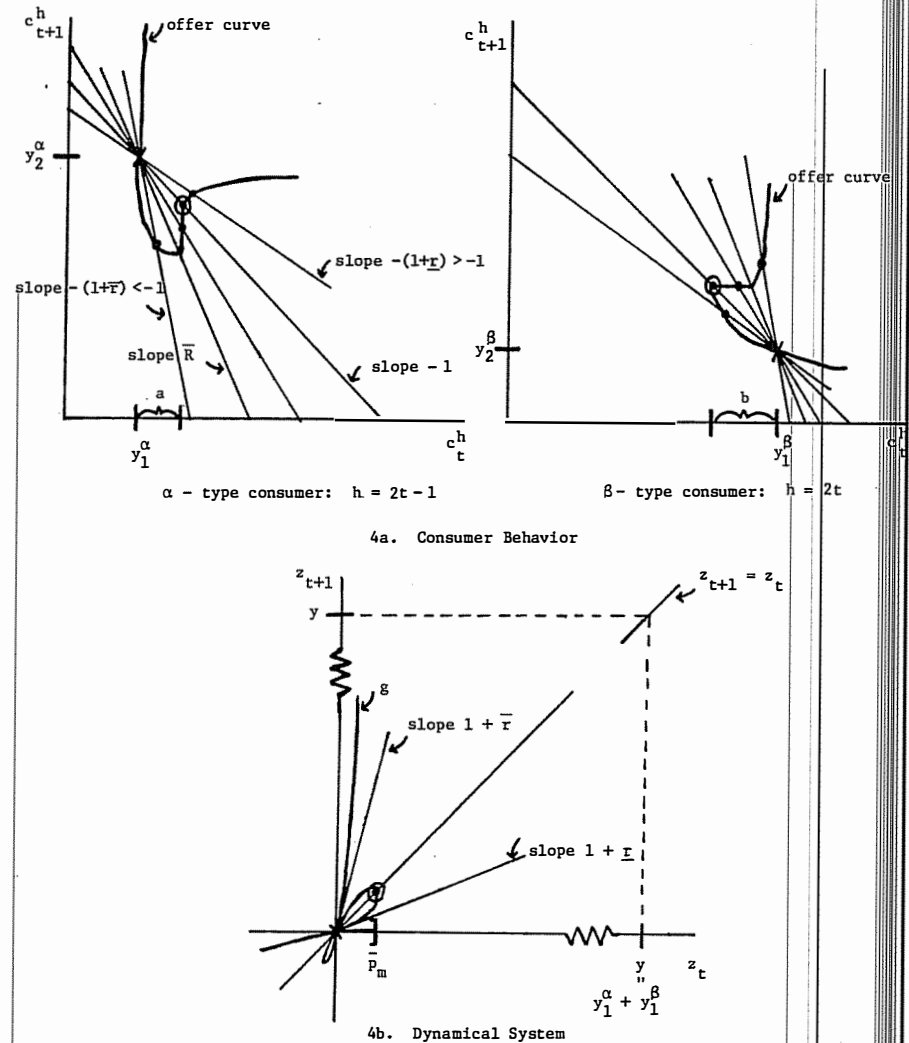
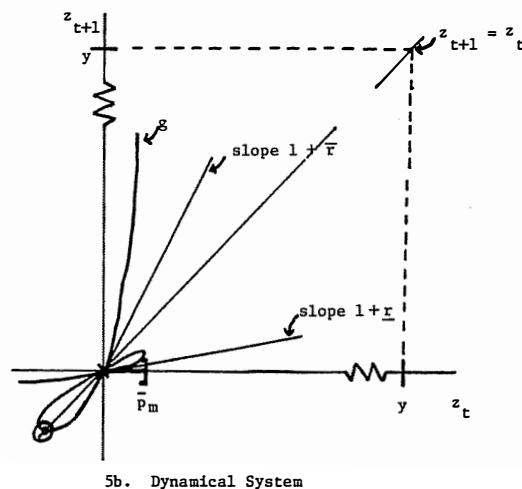
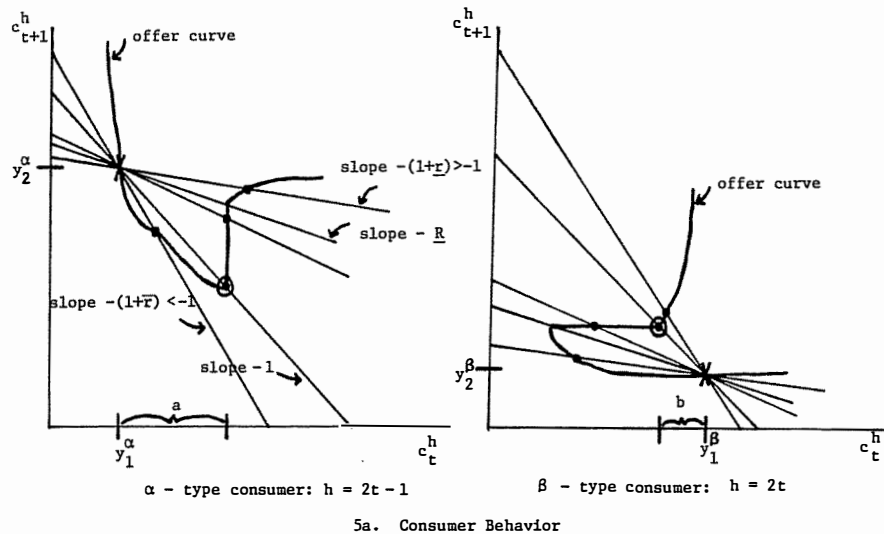


Figure 4: Coexistence with Stationary Monetary Equilibrium ( $a < b$ )

Figure 5. Coexistence with Nonstationary Monetary Equilibrium ( $a > b$ )

the  $\beta$ -type consumer wants to save more (resp., less) than the  $\alpha$ -consumer wants to dissave during their mutual first period of life, i.e., in Figure 4  $a < b$  (resp., in Figure 5  $a > b$ ), while (ii) when the real rate of return is -- within an appropriate bound -- higher,  $1 \leq p_1/p_2 \leq \bar{R}$  (resp., lower,  $\underline{R} \leq p_1/p_2 \leq 1$ ), the  $\alpha$ -type consumer has a vertical segment on his offer curve, and the  $\beta$ -type a horizontal, i.e.,  $c_t^h$  is constant for  $h = 2t - 1$ , and  $c_{t+1}^h$  for  $h = 2t$ .<sup>11</sup> Thus, in order to simplify their presentation, these specific examples embody somewhat extreme behavior, since (partly) vertical or horizontal offer curves implicitly require a region of inferiority for the good whose consumption is unchanged. Verification that this specialization is only a convenient simplification we leave as an exercise; verification that it is also a legitimate simplification we leave to the Appendix.

The critical similarity in these counterexamples is that they each display multiple stationary barter equilibria,<sup>12</sup> ranging from one which is Pareto optimal (corresponding to goods prices  $p_t = (1 + \bar{r})^{-(t-1)}$  for  $t \geq 1$ ) to one which isn't (corresponding to goods prices  $p_t = (1 + \underline{r})^{-(t-1)}$  for  $t \geq 1$ ). Their critical dissimilarity is that the first (in this sense analogous to that pictured in Figure 2) also has a single stationary monetary equilibrium -- which is, of course (by virtue of its corresponding zero real rate of return), Pareto optimal -- while the second (in this sense analogous to that pictured in Figure 1) hasn't. Reflecting on these two considerations, mainly by literally drawing their implications for the solutions to (6)

(in diagrams like Figures 4b and 5b), leads to the following important conclusions: On the one hand, in general, there is no direct connection between the level of the price of money and any descriptive or prescriptive properties of competitive equilibrium. On the other hand, in particular, there is no necessary relationship between the potentiality for monetary equilibrium and the optimality of barter equilibrium.

Moreover, the second example suggests an even more striking conclusion. Suppose we ignore the possibility of barter equilibrium (especially the possibility of any which is Pareto optimal) -- either on practical grounds (observing that monetary institutions are intrinsic to all but the most primitive societies) or, perhaps better, on theoretical grounds (appealing to the extensions of Samuelson's basic model employed to construct the examples elaborated in the last two parts of the following section). Then, this economy has the property that, though there are competitive equilibria which are Pareto optimal, (again, as in the "abnormal" case of Samuelson's basic model, essentially characterized by the condition  $\lim_{t \rightarrow \infty} \sup z_t > 0$ ), these may be neither easily discernable nor -- on any reasonable grounds -- expectedly laissez-faire. In particular, it is straightforward to establish that for large enough values of  $\bar{r} > 0$  (or  $\underline{r} < 0$ ), the smallest periodicity of any competitive equilibrium which replicates cyclically and which is also Pareto optimal may be arbitrarily large.<sup>13</sup>

In thus emphasizing the importance of periodicity (of which stationarity is the simplest realization) we are tacitly embracing the widely held professional belief that, in order for perfect foresight (or, more fashionably, rational expectations) to be a reasonable description of individual accommodation to uncertainty,

there must be sufficient regularity of aggregate outcomes -- so that, for example, correct information about the past provides a sound basis for accurate prediction of the future.

#### V. NONEXISTENCE OF ANY COMPETITIVE EQUILIBRIUM (BARTER OR MONETARY) WHICH IS PARETO OPTIMAL

Merely enlarging on Samuelson's basic model by permitting heterogeneity within each generation does not alter its central welfare implication: In such an economy, there is always some competitive equilibrium which is Pareto optimal, in fact, one which is also stationary (or, alternatively, monetary, provided that such equilibria exist). Here we present a recital of examples denying even the generality of this proposition -- counterexamples which depend, in order of their appearance at center stage, on nonstationarity in tastes and endowments (or heterogeneity across succeeding generations), nonmonotonicity in tastes (or satiation from immoderate consumption), and nonconvexity in tastes (or enhancement from unbalanced consumption).

Among these examples we can distinguish a core (specifically, the subset including only those pictured in Figures 7, 9, 10 and 11) whose significance extends far beyond their immediate bad tidings. In brief, these particular examples suggest a potentially persuasive argument in favor of continued, conscious government intervention in the competitive, market process -- one which doesn't rely on paucity of information, singularity of externalities, or any of the other standard reasons for market failure. Rather, this case rests squarely on their general, fundamental welfare implication: There may be no once-and-for-all

redistribution of wealth between the members of any finite number of generations which will permit the competitive mechanism unaided to attain a socially desirable allocation.

The validity of this assertion can be easily established simply by formalizing all such restricted wealth transfers in terms of alternative distributions of money endowments (referring for details to the comment following our description of the basic model in Section II). Now, each of the core examples, including that which depends on nonstationarity, has a very important characteristic: The particular period during which markets first open (heretofore always taken as period 1) is essentially immaterial to the behavior of the economy in that and each succeeding period -- provided that only the then current older generation has a money endowment. Moreover, given any finite distribution of money endowments, there must always come a period after which the creation (or destruction) of money ceases, or equivalently, in which -- relative to all later generations -- only the then current older generation has a money endowment. A fortiori, the demonstration of nonexistence when only the oldest generation has a money endowment suffices also when -- only to some limited extent -- other generations have as well.<sup>14</sup>

Having this point always clearly in mind, we are then well prepared to view the rehearsal of the various types of counterexamples.

#### A. Nonstationarity

The simplest stories involving nonstationarity require only slight perturbations of the most familiar example from Samuelson's

basic model, already previously sketched in Figure 2. Explicitly building on that example, suppose now that some maverick consumer  $t' \geq 1$  has either completely inflexible tastes  $U^{t'}(c^{t'}) = \min \{c_{t'}^{t'}/y_1, c_{t'+1}^{t'}/y_2\}$  or completely skewed endowments  $y^{t'} = (0, y_2)$ , while every other consumer is just as before.<sup>15</sup> Then, the dynamical system characterizing competitive equilibrium is as shown in Figure 6, which has as its only solution  $z_t = 0$  for  $t \geq 1$ . In other words, under either of these hypotheses the only competitive equilibrium is the barter one, which is obviously not Pareto optimal -- since it is possible to improve the welfare of every consumer  $t > t'$  (without affecting the welfare of any other consumer  $1 \leq t \leq t'$ ) by switching to the alternative allocation

$$c^t = \begin{cases} y^t & \text{for } 1 \leq t \leq t' \\ (y_1, y_2 + (y_1 - c_1^*)) & \text{for } t = t' + 1 \\ c^* & \text{otherwise.} \end{cases}$$

Though these examples do illustrate the essential idea underlying the whole argument in this section (namely, to structure tastes and endowments so as to bar any stationary equilibrium which is also Pareto optimal), they have at least two objectionable features. In the first place, the least complicated redistribution schemes which will permit attainment of a socially desirable allocation are all but indistinguishable from the origination of fiat money itself (albeit at some appropriate later date  $t > t'$  in the economy's history). In the second place, there is no room left for the existence of any monetary

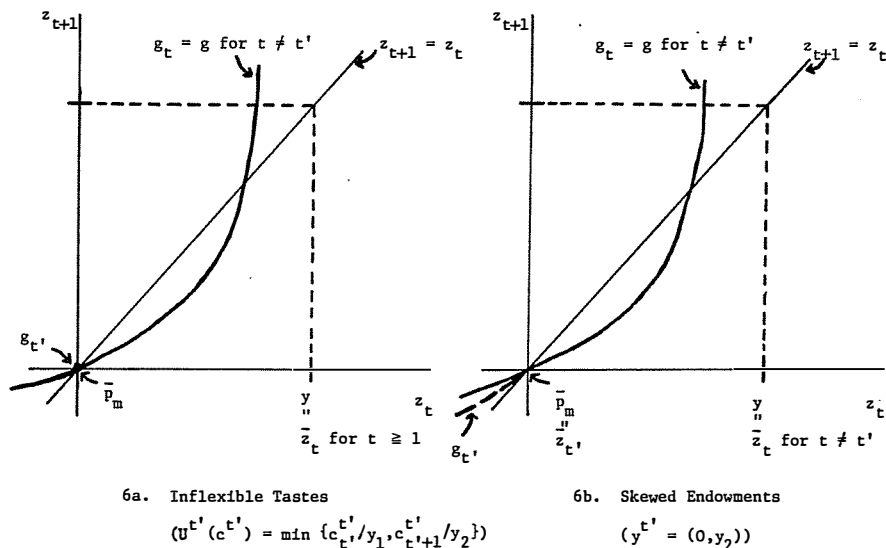


Figure 6. Nonexistence due to Nonstationarity: Two Examples  
 with a Single Jump in Tastes or Endowments

equilibrium (precisely for the same reason there is no room left for the existence of any Pareto optimal equilibrium). A necessarily more complicated example without either of these limitations -- so, in particular, with both barter and monetary equilibria -- is depicted in Figure 7. Here again we have  $G_t = \{t\}$  for  $t \geq 1$ , but now  $U^t$  is such that it (i) yields an offer curve which has a vertical segment between (3,1) and (3,4) and (ii) satisfies  $U^t(2,2) \geq U^t(3,1)$  16 while  $y^t = (3 + 3^{-(t-1)}, 1 - 3^{-t})$  for  $t \geq 1$ . Close examination of Figure 7 reveals that, although  $\bar{p}_m = 1$ , every solution to (4) (with the upper bound  $y$  suitably replaced by  $\bar{z}_t = y_t^t$ ) must satisfy

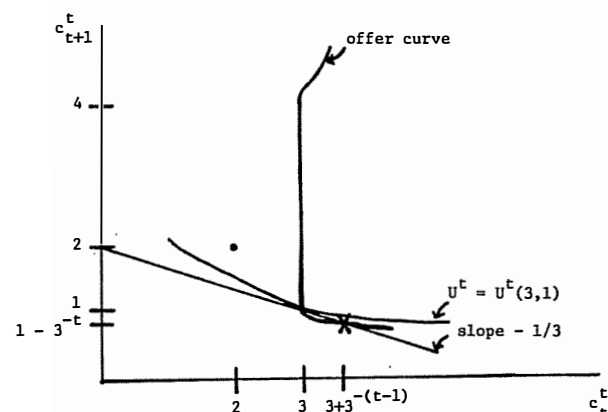
$$\frac{\partial U^t(y^t)}{\partial c_t^t} / \frac{\partial U^t(y^t)}{\partial c_{t+1}^t} \leq p_t/p_{t+1} = z_{t+1}/z_t \leq 1/3 \text{ for } t \geq 1,$$

since otherwise, typically, there will be some  $\bar{t} < \infty$  such that although  $z_{\bar{t}} < \bar{z}_{\bar{t}}$ , there is no  $z$  such that  $(z_{\bar{t}}, z) \in g_{\bar{t}}$ , which is inconsistent with (6). But from this it follows directly that every competitive allocation is strictly dominated by the stationary allocation

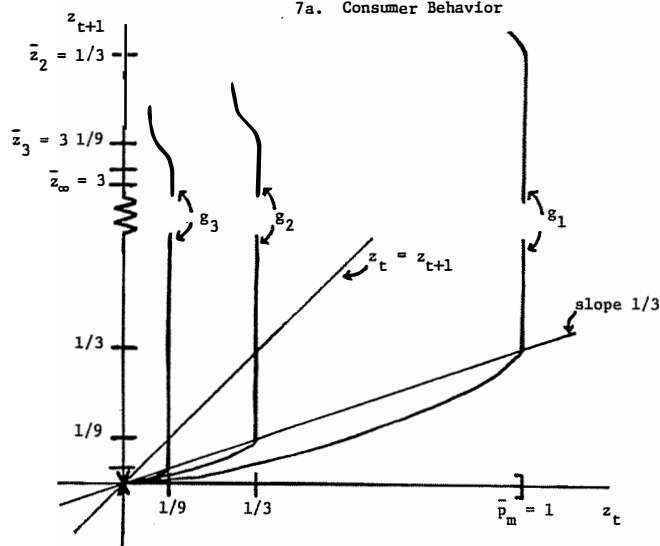
$$c^t = \begin{cases} y_1^0 + 2 & \text{for } t = 0, \\ (2, 2) & \text{otherwise} \end{cases}$$

since  $U^0(y_1^0 + 2) > U^0(y_1^0 + \bar{p}_m) \geq U^0(c_1^0)$  for every  $c_1^0 \geq 0$  such that  $c_1^0 \leq y_1^0 + \bar{p}_m$ , and if

$$\frac{\partial U^t(y^t)}{\partial c_t^t} / \frac{\partial U^t(y^t)}{\partial c_{t+1}^t} \leq p_t/p_{t+1} \leq 1/3,$$



7a. Consumer Behavior



7b. Dynamical System

Figure 7. Nonexistence due to Nonstationarity: An Example with Systematically Changing Tastes and Endowments

then

$$U^t(2,2) > U^t(3,1) \geq U^t(c_1, c_2) \text{ for every } (c_1, c_2) \geq 0$$

$$\text{such that } p_t c_1 + p_{t+1} c_2 \leq p_t y_t^t + p_{t+1} y_{t+1}^t$$

for  $t \geq 1$  (so that the welfare of every consumer is improved), while  $y_1^1 = 4$  and  $y_{t+1}^t + y_{t+1}^{t+1} = 4$  for  $t \geq 1$  (so that the allocation is feasible).

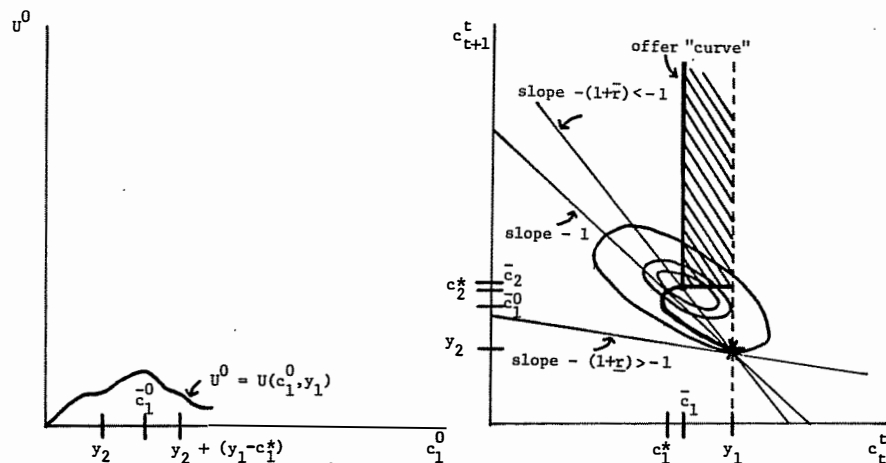
#### B. Nonmonotonicity

It is well-known that local (and hence global) satiation already raises difficulties for the first basic theorem of welfare economics within the barest standard framework usually employed in expositing the principles of general equilibrium theory. Figure 8 illustrates an example of precisely the same type of difficulty within the present context, an example again patterned after that introduced earlier in Figure 2. Here, because consumer 0 is satiated at the consumption level  $\bar{c}_1^0$  -- well below  $y_2 + (y_1 - c_1^*)$  -- every competitive allocation is dominated by one or the other of two alternatives: On the one side, if  $p_m \leq \bar{c}_1^0 - y_2$ , then the competitive allocation converges monotonically to the representative consumer's endowments, and is therefore dominated by the stationary allocation (which entails some free disposal of good 1, and hence is itself, for this reason alone, not Pareto optimal)

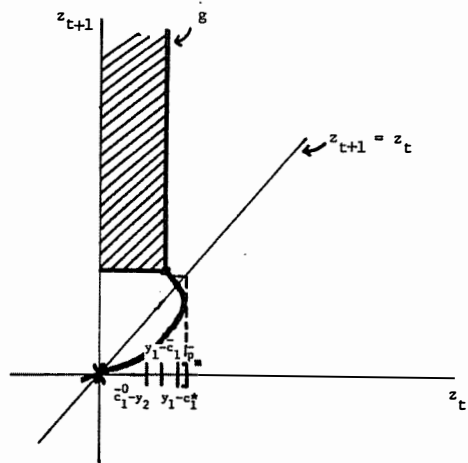
$$c^t = \begin{cases} \bar{c}_1^0 & \text{for } t = 0 \\ (c_1^*, c_2^*) & \text{otherwise.} \end{cases}$$

On the other side, if  $\bar{c}_1^0 - y_2 < p_m \leq \bar{p}_m$ , then -- because consumer 0 is





8a. Consumer Behavior



8b. Dynamical System

Figure 8. Nonexistence due to Nonmonotonicity: Global Satiation,  
Especially for the Oldest Generation

"wasting" an amount  $p_m - (\bar{c}_1^0 - y_2) > 0$  of good 1 -- it is dominated by augmenting the first period consumption of consumer 1 by some (possibly smaller) amount, everything else remaining the same (a feasible allocation which also may fail to be Pareto optimal).

Since all that this particular argument really requires is that  $U^0$  have a global maximum at a sufficiently low level of consumption, one might sensibly wonder why we've chosen to depict such an apparently unnecessarily complicated example. The sort of intergenerational symmetry consideration remarked in the second paragraph in footnote 7 provides some rationale. A more compelling justification, however, is that the general structure of tastes pictured on the righthand side of Figure 8a, and reflected in Figure 8b, suggests the possibility of constructing an example in which some lesser degree of satiation naturally puts a damper on the real rate return -- and hence the efficacy of the mere presence of money as a restorative for Pareto optimality.

In order to follow up this suggestion, we need first to introduce a second perishable physical commodity. So, as outlined previously in Section II, suppose now that there is another commodity, whose quantity is denoted  $x$ , endowment  $w$  and price  $q$ . Furthermore, suppose once more that in each generation but the oldest there are two consumers  $G_t = \{2t-1, 2t\}$  for  $t \geq 1$  whose tastes for and endowments of the two commodities depend only on whether they are odd- or even-numbered. In particular, odd-numbered consumers  $h = 2t-1$  for  $t \geq 1$  are of  $\alpha$ -type, and have tastes of the form

$$U^h(c^h, x^h) = U^\alpha(c^h) + V(x_t^h),$$

and endowments of the form

$$(y^h, w^h) = (y_1, y_2, w, 0);$$

$U^\alpha$  is differentiable, strictly quasi-concave and has a global maximum at  $(\bar{c}_1^\alpha, \bar{c}_2^\alpha)$ , where  $\bar{c}_1^\alpha < y_1$ ,  $\bar{c}_2^\alpha > y_2$  and

$$-\frac{\bar{c}_2^\alpha - y_2}{\bar{c}_1^\alpha - y_1} = 1 + \bar{r} < 1,$$

while  $V$  is differentiable, concave and strictly increasing, so that

$$\frac{dV(x)}{dx} > 0.$$

In contrast, even-numbered consumers  $h = 2t$  for  $t \geq 1$  are of  $\beta$ -type, and have both simpler tastes of the form

$$U^h(c^h, x^h) = U^\beta(c^h)$$

and simpler endowments of the form

$$(y^h, w^h) = (y_1, y_2, 0, 0);$$

$U^\beta$  is differentiable, strictly quasi-concave and strictly increasing, and also satisfies the condition

$$\frac{\partial U^\beta(y_1, y_2)}{\partial c_1} / \frac{\partial U^\beta(y_1, y_2)}{\partial c_2} = \frac{\partial U^\alpha(y_1, y_2)}{\partial c_1} / \frac{\partial U^\alpha(y_1, y_2)}{\partial c_2} = 1 + \underline{r} < 1 + \bar{r}.$$

(This last assumption merely eases description of the appropriate reflected generational offer curve.)

After modifying (2) and (3) in an obvious way to accommodate this second commodity (duly noting that all commodity prices must

still be positive in order to be consistent with market clearing), we find that for all practical purposes, this example reduces to a very special case of our basic model, by virtue of the following considerations: Only  $\alpha$ -type consumers own and value the second commodity. Hence, in every competitive equilibrium it must be true that  $x_t^{2t-1} = w$  and  $q_t > 0$  for  $t \geq 0$ . But such market clearing conditions for the second commodity will be a consequence of rational behavior on the part of  $\alpha$ -type consumers only if

$$p_t / p_{t+1} < -\frac{\bar{c}_2^\alpha - y_2}{\bar{c}_1^\alpha - y_1} = 1 + \bar{r} < 1$$

and

$$q_t = \left( \frac{dV(w)}{dx_t^{2t-1}} / \frac{\partial U^\alpha(c^{2t-1})}{\partial c_t^{2t-1}} \right) p_t$$

for  $t \geq 1$ .<sup>17</sup> In other words, in this example, the appropriate reflected generational offer curve for the purpose of characterizing the set of potential competitive equilibria is derived from the two representative consumers' offer curves under the hypothesis that the price of the second commodity can and does adjust so as to maintain each  $\alpha$ -type consumer's demand equal to his supply for that commodity. This result and its ramifications are pictured in Figure 9.

Two conclusions are immediately deducible from this figure: First, every competitive allocation necessarily converges monotonically to the representative consumers' endowments. Second, and the more significant of the two from our present perspective, every competitive allocation is thus strictly dominated by the stationary allocation

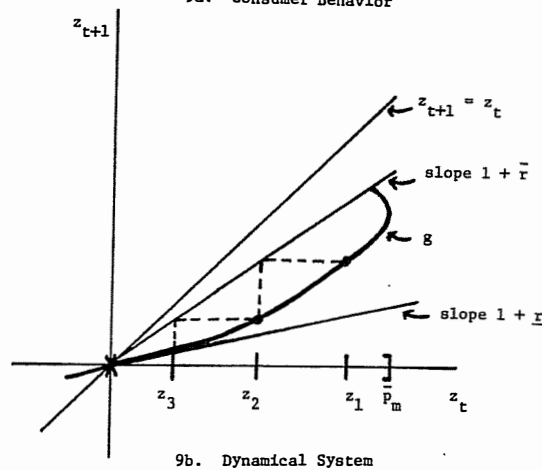
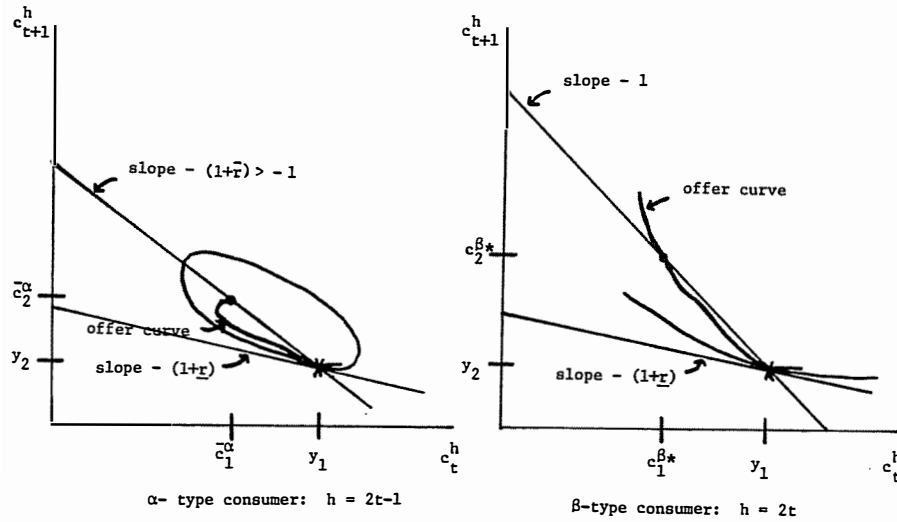


Figure 9. Nonexistence due to Nonmonotonicity: Global Satiation in the First of Two Commodities

$$c^h = \begin{cases} y_1^0 + (y_1 - \bar{c}_1^\alpha) + (y_1 - c_1^{\beta*}) & \text{for } h = 0 \\ (\bar{c}_1^\alpha, \bar{c}_2^\alpha) & \text{for } h = 2t-1 \\ (c_1^{\beta*}, c_2^{\beta*} + (y_1 - \bar{c}_1^\alpha) + (y_2 - \bar{c}_2^\alpha)) & \text{otherwise} \end{cases}$$

and

$$x^h = \begin{cases} 0 & \text{for } h = 0 \\ (w, 0) & \text{for } h = 2t-1 \\ 0 & \text{otherwise} \end{cases}$$

for  $t \geq 1$ .<sup>18</sup>

Notice especially that this example does in fact presume every consumer being locally nonsatiated, that is, always capable of increasing his utility with some arbitrarily small perturbation in his lifetime consumption profile. Thus, it clearly does not have the common feature of most familiar counterexamples to the first basic theorem of welfare economics. On the other hand, it is also clear that the example depends critically on having just the right combination of some consumption satiation together with some boundary endowments — and, for instance, this special kind of combination is well-known to create problems just for the existence of competitive equilibrium within essentially atemporal models of the allocation of consumption goods. Furthermore, since such a concatenation obviously hangs in very delicate balance, it has usually been considered somewhat of an anomaly in that context, so that one could quite rightly ask whether it should

also be so viewed in this. While we believe that the speciality of our particular counterexample (and its kin) is dictated more by our technical procedures than by our substantive objectives, this position remains to be satisfactorily buttressed. More to the point, there is a critically important distinction between the two situations being modelled; indeed, it would probably be hard to overemphasize the fact that a large measure of consumption satiation and related endowment sparsity is intrinsic to accurately portraying the essence of the intertemporal allocation of consumption goods -- both because individuals are inherently finite-lived, and because their lives are naturally several-staged.<sup>19</sup>

An even more extreme degree of "satiation cum sparsity" can be utilized to model an economy in which, though there is a stable, stationary monetary equilibrium, there is still no competitive equilibrium which is Pareto optimal. In outline, this counterexample runs as follows: Suppose now that in each generation but the oldest there are three consumers  $G_t = \{3t-2, 3t-1, 3t\}$  for  $t \geq 1$  whose tastes and endowments are described by

$$U^h(c^h, x^h) = \begin{cases} U^\alpha(c_{t+1}^h, x_t^h) \\ U^\beta(c_t^h, c_{t+1}^h) \text{ and } (y^h, w^h) = (y_1^\beta, y_2^\beta, 0, 0) \text{ for } h = 3t - 1 \\ U^\gamma(x_t^h) \end{cases} \quad (0, y_2^\alpha, w_1^\alpha, w_2^\alpha) \text{ for } h = 3t - 2$$

$$(0, 0, w^\gamma, 0) \text{ otherwise,}$$

where  $U^\alpha(c_{t+1}^h, x_t^h) = \min \{c_{t+1}^h, x_t^h\}$  and  $(0, y_2^\alpha, w_1^\alpha, w_2^\alpha)$  satisfies  $w_1^\alpha + w_2^\alpha - y_2^\alpha > 0$  but  $\approx 0$ <sup>20</sup>,  $U^\beta$  is differentiable, strictly quasi-concave and strictly increasing, and also satisfies the condition

$$\frac{\partial U^\beta(y_1^\beta, y_2^\beta)}{\partial c_1} / \frac{\partial U^\beta(y_1^\beta, y_2^\beta)}{\partial c_2} = 1 + r^\beta \approx 0,$$

and  $U^\gamma$  is (like  $U^0$ ) continuous and strictly increasing. Also, to simplify exposition, suppose that the oldest generation's endowment includes the second commodity in amount  $w_2^\alpha$  (so that  $p_m$  must be reinterpreted as the present value of 1 unit of money together with  $w_2^\alpha$  units of the second commodity).

For such an economy, after making necessary amendments to (1)-(3), it is easily seen that (i) the  $\gamma$ -type consumers' tastes entail  $q_t > 0$  for  $t \geq 1$ , while their endowments entail  $x_t^h = w^\gamma$  for  $h = 3t$ ,  $t \geq 1$ , so that (ii) the  $\alpha$ -type consumers' tastes and endowments entail both

$$c_{t+1}^h = x_t^h = w_1^\alpha + w_2^\alpha > y_2^\alpha$$

and

$$q_{t+1} = q_t + \frac{w_1^\alpha + w_2^\alpha - y_2^\alpha}{w_2} p_t > q_t$$

for  $h = 3t - 2$ ,  $t \geq 1$ . (All this, of course, based on the provisional supposition that the economy is in competitive equilibrium.) The crucial upshot of these implications is that the appropriate reflected generational offer curve now has a very special form, since the representative  $\alpha$ -type consumer's offer curve is just a single point -- independent of all commodity prices -- while the representative  $\beta$ -type consumer's offer curve is just the same as in Samuelson's basic model. Hence, the set of potential competitive equilibria can be represented as shown in Figure 10 (which explicitly assumes the least complicated, but yet still nonempty possibility, that there are only two stationary monetary equilibria). From this figure it is evident that every solution to (6) must satisfy

$$\frac{z_{t+1} - (w_1^\alpha + w_2^\alpha - y_2)}{z_t} = p_t / p_{t+1} \leq 1 + \bar{r}^* < 1,$$

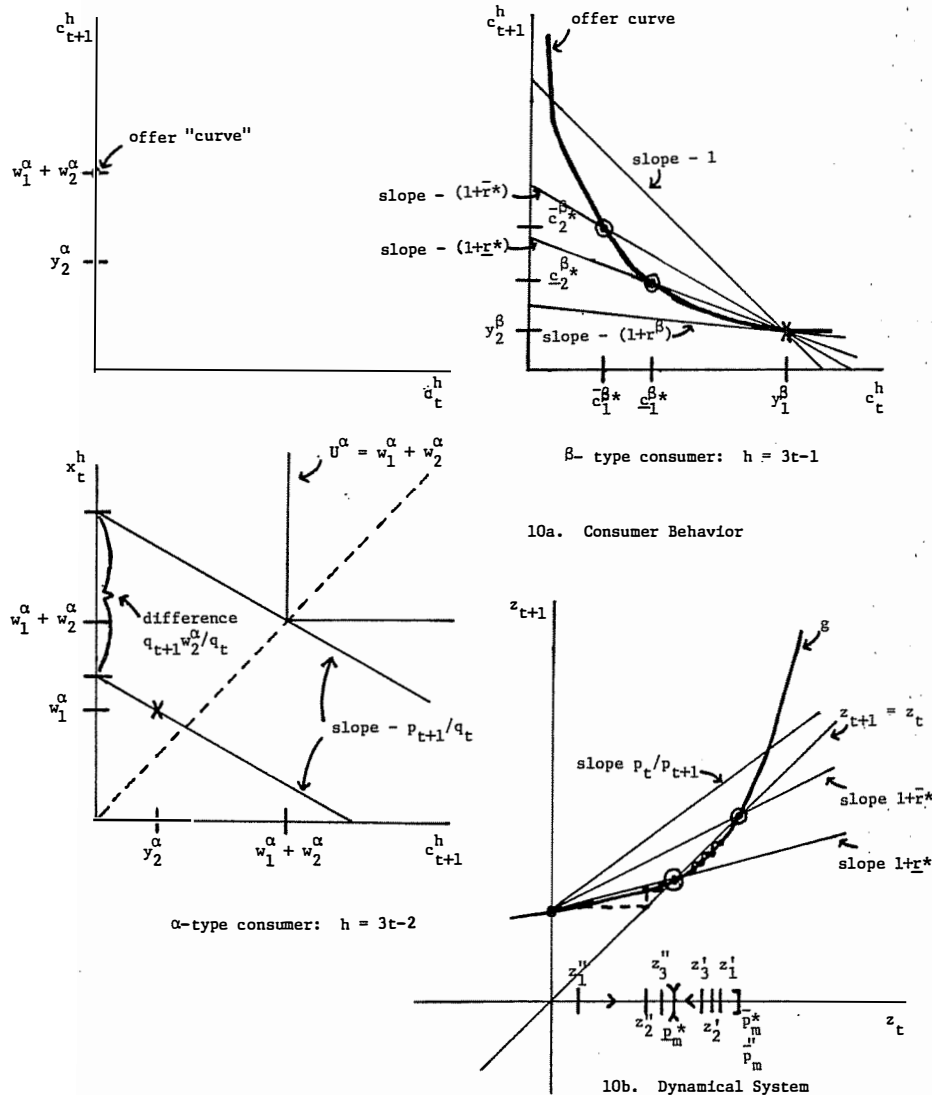


Figure 10. Nonexistence due to Nonmonotonicity: Local Satiation in the First of Two Commodities

and hence, in fact, that every competitive equilibrium either coincides with the (unstable) stationary monetary equilibrium corresponding to (the first) commodities prices

$$p_t = (1 + \bar{r}^*)^{-(t-1)} \text{ for } t \geq 1,$$

or, typically, converges to the (stable) stationary monetary equilibrium corresponding to (the first) commodities prices

$$p_t = (1 + \underline{r}^*)^{-(t-1)} \text{ for } t \geq 1.$$

Thus, it is also immediately apparent that every competitive allocation is therefore dominated, since every sequence of one-for-one forward transfers of the first commodity between only  $\beta$ -type consumers which satisfies the bounds

$$0 < \Delta c_{t+1}^{3(t-1)} = -\Delta c_t^{3(t+1)} = \Delta c_t^\beta < \bar{c}_1^\beta \text{ for } t \geq 1$$

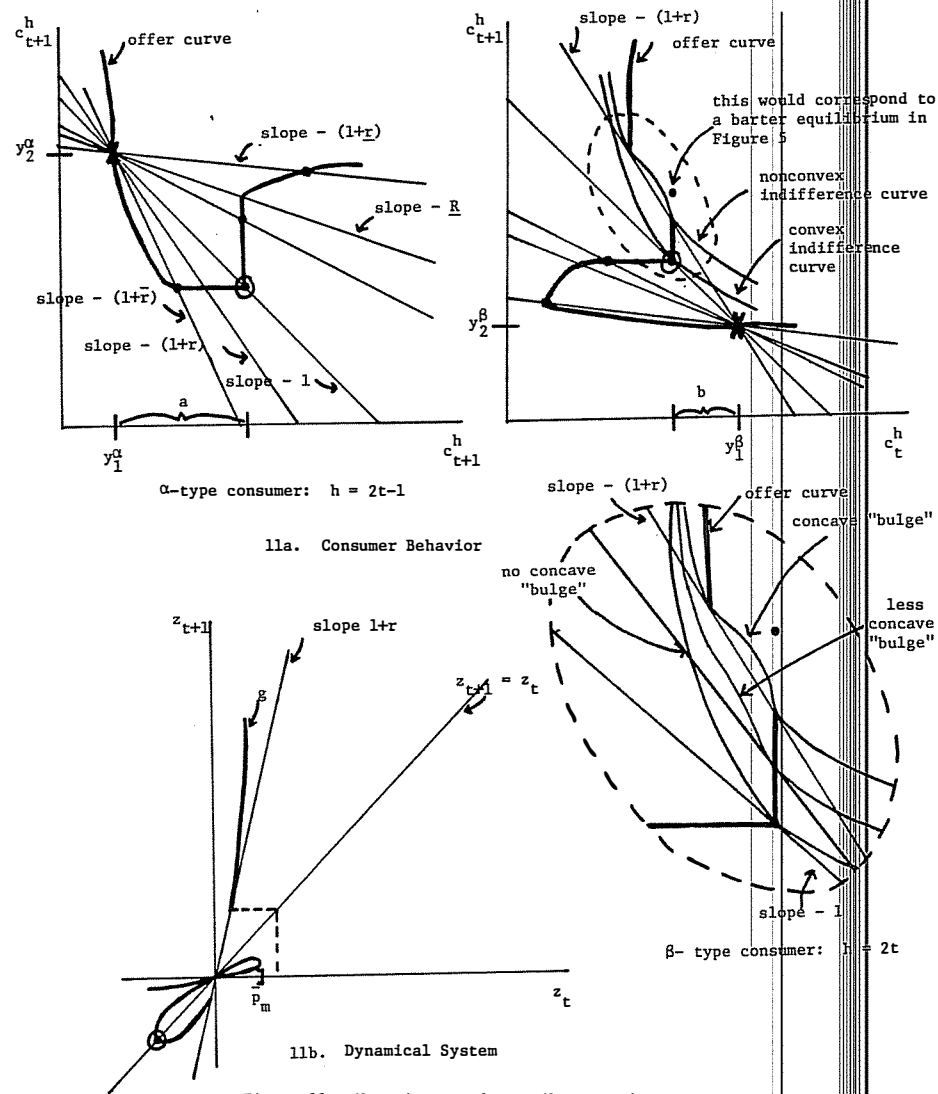
is both feasible and -- from each of their viewpoints (by virtue of the fact that at best they face a uniformly negative real rate of return  $p_t/p_{t+1} - 1 \leq \bar{r}^* < 0$  for  $t \geq 1$ ) -- preferable.

### C. Nonconvexity

This counterexample is nothing more than a straightforward variation of the example presented previously in Figure 5, and beyond simply displaying its structure -- as we do in Figure 11 -- requires only justifying its construction -- as we do in the Appendix.

Note, however, that this same device, namely, introducing some nonconvexity into the upper reaches of the  $\beta$ -type consumer's

indifference map, can also be employed to provide a solid theoretical foundation for our earlier emphasis on the inherent difficulties attendant on nonuniqueness of monetary equilibrium (at the end of Section IV). The doggedly perservering reader should, by now, be able to follow the dots.



## FOOTNOTES

1. Some asymmetry in treating the start of the economy is unavoidable, since, for example, consumers in the oldest generation have only themselves to deal with during their first period of life. For this reason we will typically streamline the various incarnations of our basic model by simply assuming that the oldest generation consists of just a single consumer. It can be easily verified that nothing we have to say depends critically on this particular simplification.

We should also emphasize at the outset that, unlike most of the literature, our specializations of the consumption-loan model will always involve a stationary population (at least after period 1). Once again, nothing depends critically on this particular simplification -- and it has the great virtue of completely avoiding the notational clutter inevitably associated with modelling a growing population.

2. Except in one case (in the second subsection of V), we always maintain that these properties obtain for the tastes of consumers in the oldest generation. Since the relevant aspect of their lifetime consumption profiles is a single quantity, this simply means that they have utility functions which are continuous and strictly increasing.

Likewise, we will almost invariably assume that, for every good, there is some consumer who has a utility function which is everywhere

strictly increasing in that good. The practical import of this hypothesis is that in a competitive equilibrium, every goods price will be positive (see the definition in the next following paragraph, and the subsequent discussion of the characterization of potential competitive equilibria).

3. Generally, we should write the price of money  $p_{mt}$ . However, if we expand the budget constraint in (2) below to reflect such opportunities,

$$\begin{cases} p_t c_t^h + p_{t+1} c_{t+1}^h + p_{mt} \Delta m_t^h + p_{mt+1} \Delta m_{t+1}^h \leq p_t y_t^h + p_{t+1} y_{t+1}^h, \\ \Delta m_t^h \geq 0 \\ \Delta m_{t+1}^h \geq -\Delta m_t^h \end{cases}$$

where  $\Delta m_s^h$  represents the purchases (or sales, when negative) of money during period  $s = t, t+1$  by consumer  $h$ , then an obvious arbitrage argument entails that, in a competitive equilibrium,  $p_{mt} = p_m$  for  $t \geq 1$ . For this reason we have chosen simply to adopt this requirement as a postulate -- and correspondingly (and legitimately) to ignore transactions on the money markets except insofar as they affect the oldest generation's demand for goods.

4. This maneuver presumes that  $y_t^h > 0$  for some  $h \in G_t$  for all  $t \geq 1$ . Otherwise, the upper bound in (6) must be written  $\bar{z}_t$ , a "0-y" variable.

We will refer to such a polar situation only once in the sequel (see the second example of nonstationarity, that involving only endowments, presented in Section V below). In one other case (the third example in the same subsection) we will deliberately choose to streamline the description of a specific model by explicitly employing "natural" units for measuring the good in each period, so that the upper bound in (6) must also be written generally as  $\bar{z}_t = \sum_{h \in G_t} y_t^h$ .

5. It should be immediately apparent from the definitions of  $z_t$  and  $g_t$  that a competitive equilibrium yields a solution to (6). On the other hand, the reverse argument only requires observing that, in the definition of  $g_t$ , goods prices can be normalized so that they constitute a consistent sequence -- since they are assumed positive, while consumers are assumed rational (so that, among other things, those in each generation  $t \geq 2$  correctly perceive that they are unaffected by equi-proportional shifts in the goods prices directly relevant to them  $(p_t, p_{t+1})$ ).
6. When  $z_t = 0$  for  $t \geq 1$ , this equation amounts to a definition of  $0/0$  (and it is obviously necessary to go back to the basic data to uncover the potential paths of the real rate of return  $r_t$  for  $t \geq 1$ ).
7. In this case, as in some others later on, there is no need to distinguish  $h$  from  $t$ , except insofar as the former appears as a

superscript, the latter as a subscript.

It is also worth remarking that consumer 0's tastes and endowments (of goods) can also be assumed the same as any consumer  $t$ 's, merely by reinterpreting  $U^0(c_1^0) = U(c_1^0, y_1^0)$  and  $y_1^0 = y_2$ . More generally, in each of the special cases we treat later on in which both tastes and endowments are essentially stationary, we can reinterpret the model so that generation 0 is identical to generation  $t \geq 1$  but for date of birth and absence of ancestors (or, more accurately, ancestors who participate in the market economy). The possibility of such reinterpretation provides the main justification for the assertion made at the end of the first paragraph in footnote 1.

8. A note of warning: Here, as in the sequel, we take for granted that every solution to (6) must satisfy  $z_1 \leq \bar{p}_m \leq y$  -- where  $\bar{p}_m$  depends on both  $g_t$  and  $y$  in a way which should be obvious from the diagram representing the relevant dynamical system. That is, otherwise, when  $z_1 > \bar{p}_m$ , there will be some  $\bar{t} < \infty$  such that either  $z_{\bar{t}} > y$  or  $z_{\bar{t}} \leq y$  but there is no  $z$  such that  $(z_{\bar{t}}, z) \in g_{\bar{t}}$ , both of which are inconsistent with (6).
9. The artistically inclined reader can surely sketch examples with much longer periodicity -- once he is reminded that the only significant restrictions on  $g$  are that it be continuous (where the independent variable is  $p_1/p_2 > 0$ ), that it increase in at least one element (with increases in  $p_1/p_2$ ), and that it intersect each positively sloped ray through the origin, but just once. (See too



the introductory comments in the Appendix). We should also note that our labelling of this second case is not to be taken seriously (paranoically?). The "abnormal" case, which was apparently first mentioned in the literature by Gale, seems just as likely as the "normal" case -- even if it is much more bothersome, because it so vividly displays the ambiguity inherent in the actual evolution of monetary equilibrium. We shall have more to say about the implications of this characteristic sort of nonuniqueness in the following section.

10. These results (and their parallels in the subsequent sections) are so well-known that they hardly require documentation; for completeness' sake, however, we refer the skeptically minded reader to Starrett's neat general argument. Note also that such results may be falsified (in case  $1 + r \leq 1$ ) when  $U$  is not assumed to be strictly quasi-concave, as in Shell's nice expository piece.
11. The upper bound  $\bar{R}$  (resp., lower bound  $\underline{R}$ ) is chosen so that the length of the vertical and horizontal segments of the respective consumers' offer curves is greater than  $b-a$  (resp.,  $a-b$ ). This choice guarantees (see directly below) that there are at least two stationary barter equilibria with positive (resp., negative) real rate of return.
12. Since -- in consumption loan type models with only a single period

overlap between generations -- barter involves just trade between contemporaries, this also means that barter equilibrium needn't be stationary, nor even periodic (see, however, the closing remark to this section). For instance, there is an "irregular" barter equilibrium corresponding to goods prices which satisfy

$$p_{t+1}/p_t = \begin{cases} 1 + \bar{r} & \text{for } 2^s \leq t < 2^{s+1}, s \geq 0 \\ 1 + \underline{r} & \text{otherwise.} \end{cases}$$

13. Formally, periodic monetary equilibrium requires that for some finite span of periods, or periodicity  $1 \leq \tau < \infty$ , goods prices must exhibit the property that  $p_{t+\tau} = p_t$  or  $\prod_{s=0}^{\tau-1} (p_{t+s}/p_{t+s+1}) = \prod_{s=0}^{\tau-1} (1+r_{t+s}) = 1$  for  $t \geq 1$ . But in this case, any such periodicity must therefore also satisfy the inequality

$$\min_{1 \leq \tau' \leq \tau} (1 + \bar{r})^{\tau'} (1 + \underline{r})^{\tau - \tau'} = (1 + \bar{r})(1 + \underline{r})^{\tau-1} < 1,$$

from which the assertion in the text follows immediately.

14. This argument is not quite complete, since it implicitly requires that every competitive allocation can be strictly improved upon generation by generation. However, we will in fact demonstrate that this stronger form of Pareto superiority is feasible in each of the core examples.

In this context, it is an interesting problem (though not one we will pursue further here) to characterize the simplest

wealth transfers that can be (i) achieved, for instance, by assigning just nonnegative money endowments (i.e., ostensibly, by issuing subsidies rather than collecting taxes) and also (ii) instrumental in supporting the Pareto optimality of competitive equilibrium.

15. Thanks are due Charles Hulten of John Hopkins for pointing out the relevance of the first of these hypotheses for our present purpose. This hypothesis, of course, already introduces an element of satiation. Note too that, for the second hypothesis to be consistent with the existence of competitive equilibrium, we require the boundary condition

$$\frac{\partial U(0, y_2)}{\partial c_1} / \frac{\partial U(0, y_2)}{\partial c_2} < \infty.$$

16. Once again, see the Appendix for detailed instructions on how to lay out an indifference map which yields such an offer curve, and yet at the same time satisfies such an additional restriction. In fact, this example doesn't require quite such extreme behavior. The subsequent argument remains true, for instance, provided each offer curve has a segment from (3,1) to (.,4) with slope

$$\frac{dc_{t+1}^t}{dc_t^t} \text{ (or average } \frac{c_{t+1}^t - 1}{c_t^t - 3} \text{ ) greater than minus one.}$$

17. The first condition must hold because otherwise, that is, when

$$p_{t'}/p_{t'+1} \geq - \frac{\bar{c}_2^\alpha - y_2}{\bar{c}_1^\alpha - y_1} = 1 + \bar{r} \text{ for some } t' < \infty,$$

consumer  $2t' - 1$  would demand  $x_{t'}^{2t'-1} > w$  -- since under these circumstances his budget constraint encompasses the lifetime consumption profile

$$(\bar{c}_1^\alpha, \bar{c}_2^\alpha, x) \text{ with } x = w + \frac{p_{t'}(y_1 - \bar{c}_1^\alpha) + p_{t'+1}(y_2 - \bar{c}_2^\alpha)}{q_{t'}} \geq w,$$

while at this particular profile he is completely satiated in both his periods consumption of the first commodity, but not his first period consumption of the second commodity.

The second condition is then essentially just the first order requirement for the optimal solution to (2) (suitably expanded) to satisfy  $x_{t'}^{2t'-1} = w$  given the first condition.

18. This assertion explicitly requires that  $\bar{p}_m < y_1^0 + (y_1 - \bar{c}_1^\alpha) + (y_1 - c_1^{\beta*})$ , a restriction which is easily satisfied, for instance, by specifying the  $\beta$ -consumer's choice of first period consumption at the real rate of return 0 small enough relative to that same choice at every real rate of return  $r \leq \bar{r} < 0$ , everything else unchanged.
19. In this connection, it is also worth remarking that we have developed variants of the counterexample depicted in Figure 9 which are grounded only on having heterogeneity across generations -- but heterogeneity which is repeated regularly (so that, for

instance, generations are alternately "thrifty" and "spend-thrifty") -- and in which it is inevitable that some consumers possess the only endowments of some goods which they alone value.

We have chosen to present the more artificial construct since these alternatives necessarily involve introducing a third stage of the life cycle, and since introducing a specialized second commodity also admits yet another variety of quite interesting counterexample (as we shall now proceed to demonstrate).

20. The  $\alpha$ -type consumers could be assumed to have more flexible tastes; all that is required in the following argument is that their choices of second period consumption satisfy the lower bound  $c_{t+1}^h \geq y_2^\alpha + \epsilon$  for  $t \geq 1$ , for some fixed positive number  $\epsilon > 0$ .

Here and below " $\approx 0$ " means something like "small enough so that Figure 10 is qualitatively accurate."

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## APPENDIX: Construction of Vertical or Truncated Offer Curves

### AI. Introduction and Background

The purpose of this appendix is to substantiate the several claims made in the text (both explicitly and implicitly) regarding the possibility of constructing various types of individual offer curves. In accomplishing this purpose, it essentially amounts to an elementary exercise in demand theory.

The following central result is well-known: Consider the typical consumer in our basic model, and suppose that his utility function  $U^h$  is continuous, strictly increasing and strictly quasi-concave, while his endowment vector  $(y_t^h, y_{t+1}^h)$  is nonnegative and nontrivial. Then, if  $(c_t^h, c_{t+1}^h)$  represents the unique optimal solution to his budget constrained utility maximization problem (2) as depending on (positive, finite) relative prices -- or, in common parlance, his demand functions -- these are (nonnegative and) continuous and satisfy his (relative price) budget constraint

$$(p_t/p_{t+1})c_t^h + c_{t+1}^h = (p_t/p_{t+1})y_t^h + y_{t+1}^h.$$

Moreover, even if  $U^h$  has a strict global maximum, say, at  $(\bar{c}_t^h, \bar{c}_{t+1}^h)$  (but is elsewhere strictly increasing or decreasing), the same result remains true provided, in addition, either his endowment vector is weakly dominated,  $(y_t^h, y_{t+1}^h) \leq (\bar{c}_t^h, \bar{c}_{t+1}^h)$ , or he is not satiated at his endowment vector,  $(y_t^h, y_{t+1}^h) \neq (\bar{c}_t^h, \bar{c}_{t+1}^h)$ , and relative prices are limited in range,

$$p_t/p_{t+1} \left\{ \begin{array}{l} \leq \\ \geq \end{array} \right\} - \frac{\bar{c}_{t+1}^h - y_{t+1}^h}{\bar{c}_t^h - y_t^h} \quad \text{according as } \bar{c}_t^h \left\{ \begin{array}{l} < \\ > \end{array} \right\} y_t^h.$$

The converse to these propositions is already falsified by simple examples of the sort pictured in Figures 4 and 5 (assuming their validity, which follows from an argument similar to that presented explicitly in the next section). In particular, under either alternative set of maintained assumptions, the optimal consumption vector will coincide with the endowment vector for at most a single relative price (except in the singular situation where  $(y_t^h, y_{t+1}^h) = (\bar{c}_t^h, \bar{c}_{t+1}^h)$ , in which case they obviously coincide at every relative price). Hence, it would be accurate to say that the general problem (formulated in terms of specific questions) we are addressing here is: To what extent does the property of representing the demand functions  $(c_t^h, c_{t+1}^h)$  -- or, equivalently, the excess demand functions  $(c_t^h - y_t^h, c_{t+1}^h - y_{t+1}^h)$  composing the offer curve, for short, simply the offer curve -- impose further restrictions beyond just continuity and satisfaction of a budget constraint?

In a very elegant development initiating with Sonnenschein [12,13] and culminating with Mantel [9] and Debreu [4], it has recently been established that -- under analogous maintained assumptions for  $n \geq 2$  commodities and  $m \geq n$  individuals -- aggregate excess demand functions are completely characterized by continuity (in uniformly positive simplicial prices) and Walras' law. While this fundamental result obviously has some indirect bearing on the topic of this essay, especially, on the conclusions exemplified by Figures 4 and 5, even in these examples it is not decisive for our purposes. Indeed, from the main line of argument in the text it should be clear that both continuity and (the analogue of) Walras' law play crucial roles in

delimiting the set of potential competitive equilibria in our basic model. Thus, for instance, the former rules out the possibility that there is no Pareto optimal stationary equilibrium, the latter that there are multiple stationary monetary equilibria in Section IV.

Moreover, it almost goes without saying that the requirement of market clearing is always at center stage in our presentation. This condition is especially important, for instance, in ruling out the possibility that there is some Pareto optimal stationary equilibrium -- and, hence, any Pareto optimal competitive equilibrium -- in Section V.

### AII. Vertical Offer Curves

We concentrate attention on the details of the nonstationarity example described in Figure 7. Also, in order to simplify the discussion, here we will employ somewhat more conventional notation, namely,  $(x, y)$  for the consumption vector, (again)  $U$  for the utility function and  $(\bar{x}, \bar{y})$  for the endowment vector. Exactly the same basic principles apply to rationalizing the preference structure underlying the heterogeneity examples described in Figures 4 and 5 -- except that, roughly speaking, the roles of  $x$  and  $y$  become reversed.

What we propose to show, then, is that given (i)  $(\underline{x}, \underline{y}) > 0$  such that  $\underline{x} < \bar{x}$  and  $\bar{y} > \underline{y}$ , (ii)  $\bar{y} > \underline{y}$  and (iii)  $(x^*, y^*) > 0$  such that  $x^* < \underline{x}$  and  $y^* > \underline{y}$ , it is possible to construct  $U : R_+^2 \rightarrow R$  which is continuous, strictly increasing, strictly quasi-concave and such that

1. The offer curve originating at  $(\bar{x}, \bar{y})$  is vertical between  $(\underline{x}, \underline{y})$  and  $(\bar{x}, \bar{y}) \equiv (\underline{x}, \bar{y})$ .<sup>#</sup>

<sup>#</sup> Thus, for the sake of symmetry in various expressions, we will interchangeably use the notation  $\bar{x} = \underline{x}$ , and later on,  $x' = \underline{x}$  as well.

2. The indifference curve passing through  $(x, y)$  lies strictly below (i.e., to the southwest of)  $(x^*, y^*)$ .

The idea of this construction is wholly geometric, and is fully elucidated by Figure A1. An algorithm for the construction proceeds as follows: Pick two functions  $\underline{f} : R_+ \rightarrow R$  and  $\bar{f} : R_+ \rightarrow R$  which are differentiable, strictly decreasing and strictly convex, which satisfy  $\underline{f}(x) < \bar{f}(x)$  for  $x \geq 0$  and which yield  $\underline{f}(x^*) < y^*$ ,  $\underline{f}(x) = \underline{y}$ ,  $\underline{f}'(x) = \frac{\underline{y} - \tilde{y}}{x - \tilde{x}}$  and  $\bar{f}(x) = \bar{y}$ ,  $\bar{f}'(x) = \frac{\bar{y} - \tilde{y}}{x - \tilde{x}}$ , respectively. In particular, this choice entails that the curves  $y = \underline{f}(x)$  and  $y = \bar{f}(x)$  for  $x \geq 0$  have the following relationships to the budget constraints

$$(A1) \quad \begin{cases} -\frac{\underline{y} - \tilde{y}}{x - \tilde{x}} x + y = -\frac{\underline{y} - \tilde{y}}{x - \tilde{x}} \tilde{x} + \tilde{y} \text{ or } y = \frac{\tilde{y}x - \underline{y}\tilde{x}}{x - \tilde{x}} + \frac{\underline{y} - \tilde{y}}{x - \tilde{x}} x \\ \text{and} \\ -\frac{\bar{y} - \tilde{y}}{x - \tilde{x}} x + y = -\frac{\bar{y} - \tilde{y}}{x - \tilde{x}} \tilde{x} + \tilde{y} \text{ or } y = \frac{\tilde{y}x - \bar{y}\tilde{x}}{x - \tilde{x}} + \frac{\bar{y} - \tilde{y}}{x - \tilde{x}} x, \end{cases}$$

respectively:

$$(A2) \quad \begin{cases} \underline{f}(x) \begin{cases} > \\ = \end{cases} \frac{\tilde{y}x - \underline{y}\tilde{x}}{x - \tilde{x}} + \frac{\underline{y} - \tilde{y}}{x - \tilde{x}} x \text{ according as } x \begin{cases} \neq \\ = \end{cases} \tilde{x} \\ \text{and} \\ \bar{f}(x) \begin{cases} > \\ = \end{cases} \frac{\tilde{y}x - \bar{y}\tilde{x}}{x - \tilde{x}} + \frac{\bar{y} - \tilde{y}}{x - \tilde{x}} x \text{ according as } x \begin{cases} \neq \\ = \end{cases} \tilde{x}. \end{cases}$$

Generally, such functions  $\underline{f}$  and  $\bar{f}$  can be found in the parametric class

$$f(x) = a + be^{\beta x} + ce^{\gamma x} \text{ with } a + b + c > 0, a < 0, (b, c) > 0 \text{ and } (\beta, \gamma) < 0.$$

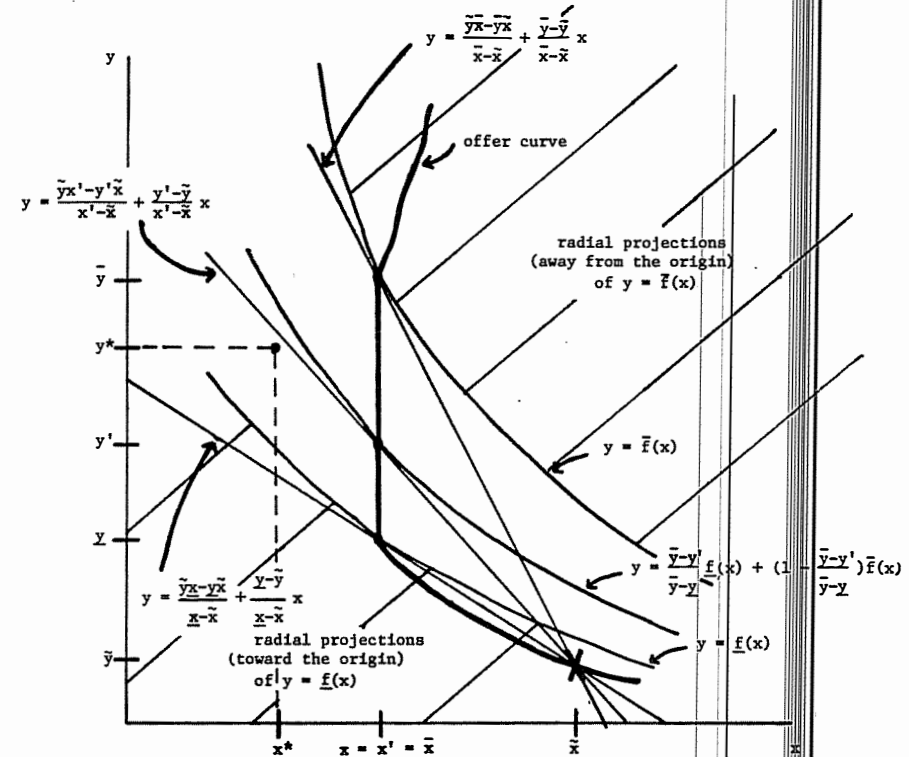


Figure A1. Construction of a Utility Function which Yields a Vertical Segment on the Offer Curve and Satisfies an Additional Dominance Condition.

So, without any loss of generality, assume in addition that  $\lim_{x \rightarrow \infty} \underline{f}(x) = \lim_{x \rightarrow \infty} \bar{f}(x) < 0$ , so that, in particular, both functions also intersect the x-axis.

Next, define indifference curves (covering the whole nonnegative quadrant) in terms of the two functions  $\underline{f}$  and  $\bar{f}$  thusly:

(i) for  $(x', y') \geq 0$  and  $0 \leq y' \leq \underline{f}(x')$ , the indifference curve is the appropriate segment (i.e., that lying in the nonnegative quadrant) of the radial projections of  $(x, \underline{f}(x))$  (i.e., as  $x$  varies over the nonnegative halfline) toward the origin in proportion

$$(A3) \quad \alpha = \begin{cases} 0 & \text{if } x' = y' = 0 \\ y'/\underline{f}(x') & \text{if } x' = 0, y' > 0 \\ x'/x'' & \text{otherwise,} \end{cases}$$

where  $x''$  is defined by  $\underline{f}(x'')/x'' = y'/x'$  if  $x' > 0$ ; (ii) for  $(x', y') \geq 0$  and  $\underline{f}(x') \leq y' \leq \bar{f}(x')$ , the indifference curve is the appropriate segment of the convex combinations of  $(x, \underline{f}(x))$  and  $(x, \bar{f}(x))$  using weights

$$(A4) \quad \alpha = \frac{\bar{f}(x') - y'}{\bar{f}(x') - \underline{f}(x')} \text{ and } 1 - \alpha = 1 - \frac{\bar{f}(x') - y'}{\bar{f}(x') - \underline{f}(x')},$$

respectively; and (iii) for  $(x', y') \geq 0$  and  $y' \geq \bar{f}(x')$ , the indifference curve is the appropriate segment of the radial projections of  $(x, \bar{f}(x))$  away from the origin in proportion

$$(A5) \quad \alpha = \begin{cases} y'/\bar{f}(x') & \text{if } x' = 0, y' > 0 \\ x'/x'' & \text{otherwise,} \end{cases}$$

where now  $x''$  is defined by  $\bar{f}(x'')/x'' = y'/x'$  if  $x' > 0$ .

Finally, simply label each indifference curve with the value of its y-intercept.

Establishing that the utility function so constructed has the requisite continuity, monotonicity and convexity properties is a routine matter. To begin with, observe that, given our particular choice of labelling for the indifference curves, in each of the three regions delineated by  $\underline{f}$  and  $\bar{f}$  the various definitions of (A3)-(A5) can be recast in terms of  $U$ ,  $\underline{f}(0)$  and  $\bar{f}(0)$ . For instance, (A4) is equivalent to

$$(A4') \quad \alpha = \frac{\bar{f}(0) - U}{\bar{f}(0) - \underline{f}(0)} \text{ and } 1 - \alpha = 1 - \frac{\bar{f}(0) - U}{\bar{f}(0) - \underline{f}(0)},$$

where  $U$  is related to  $(x', y')$  implicitly by the equation

$$(A6) \quad y' = \frac{\bar{f}(0) - U}{\bar{f}(0) - \underline{f}(0)} \underline{f}(x') + \left(1 - \frac{\bar{f}(0) - U}{\bar{f}(0) - \underline{f}(0)}\right) \bar{f}(x')$$

Following this lead through, it is easy to show that, in general,  $U$  is defined implicitly in terms of its indifference curves by the formulae

$$(A7) \quad y = \begin{cases} (U/\underline{f}(0))\underline{f}(x) & \text{for } x \geq 0, \underline{f}(x)/\underline{f}(0) \geq U/\underline{f}(0) \text{ and } 0 \leq U \leq \underline{f}(0) \\ \frac{\bar{f}(0) - U}{\bar{f}(0) - \underline{f}(0)} \underline{f}(x) + \left(1 - \frac{\bar{f}(0) - U}{\bar{f}(0) - \underline{f}(0)}\right) \bar{f}(x) & \text{for } x \geq 0, (\bar{f}(0) - U)\underline{f}(x) + (U - \underline{f}(0))\bar{f}(x) \geq 0 \text{ and } \underline{f}(0) \leq U \leq \bar{f}(0) \\ (U/\bar{f}(0))\bar{f}(x) & \text{for } x \geq 0, \bar{f}(x)/\bar{f}(0) \geq U/\bar{f}(0) \text{ and } \bar{f}(0) \leq U. \end{cases}$$

But by virtue of the specification of  $\underline{f}$  and  $\bar{f}$ , each of the functions on the righthand side of (A7) is differentiable, strictly decreasing in  $x$ , strictly increasing in  $U$  and strictly convex in  $x$  on its respective domain. Hence, the desired properties follow directly upon application of well-known, elementary results from functional analysis.

Utilizing yet another simple reformulation of (A4), it is also easy to verify the first additional property motivating this whole exercise -- that the offer curve originating at  $(\tilde{x}, \tilde{y})$  is vertical between  $(\underline{x}, \underline{y})$  and  $(\bar{x}, \bar{y})$ . In particular, from (A6) we see that

$$\frac{\bar{f}(0) - U}{\bar{f}(0) - \underline{f}(0)} = \frac{y' - \bar{f}(x')}{\underline{f}(x') - \bar{f}(x')} = \frac{\bar{f}(x') - y'}{\bar{f}(x') - \underline{f}(x')}.$$

Thus, by fixing  $x' = \underline{x} = \bar{x}$  (so that, as in Figure A1,  $\underline{f}(x') = \underline{f}(\underline{x}) = \underline{y}$  and  $\bar{f}(x') = \bar{f}(\bar{x}) = \bar{y}$ ), and considering only the nonnegative region lying between the prespecified curves  $y = \underline{f}(x)$  and  $y = \bar{f}(x)$  for  $x \geq 0$ , the indifference curves described in (A7) can equally well be described by

$$(A8) \quad y = \frac{\bar{y} - y'}{\bar{y} - \underline{y}} \underline{f}(x) + \left(1 - \frac{\bar{y} - y'}{\bar{y} - \underline{y}}\right) \bar{f}(x) \text{ for } x \geq 0, \underline{y} \leq y' \leq \bar{y}.$$

Since the budget constraint

$$-\frac{y' - \tilde{y}}{x' - \tilde{y}} x + y = -\frac{y' - \tilde{y}}{x' - \tilde{y}} \tilde{x} + \tilde{y} \text{ or } y = \frac{\tilde{y}x' - y'\tilde{x}}{x' - \tilde{x}} + \frac{y' - \tilde{y}}{x' - \tilde{x}} x$$

is identical to that obtained by taking the same convex combination of the two budget constraints described in (A1), that is,

$$y = \left(\frac{\bar{y} - y'}{\bar{y} - \underline{y}}\right) \left(\frac{\tilde{y}x' - y'\tilde{x}}{x' - \tilde{x}} + \frac{y' - \tilde{y}}{x' - \tilde{x}} x\right) + \left(1 - \frac{\bar{y} - y'}{\bar{y} - \underline{y}}\right) \left(\frac{\tilde{y}x' - y'\tilde{x}}{x' - \tilde{x}} + \frac{y' - \tilde{y}}{x' - \tilde{x}} x\right) \\ = \frac{\tilde{y}x' - y'\tilde{x}}{x' - \tilde{x}} + \frac{y' - \tilde{y}}{x' - \tilde{x}} x,$$

(A8) together with (A2) immediately entail

$$(A9) \quad \frac{\bar{y} - y'}{\bar{y} - \underline{y}} \underline{f}(x) + \left(1 - \frac{\bar{y} - y'}{\bar{y} - \underline{y}}\right) \bar{f}(x) \begin{cases} > \\ = \\ < \end{cases} \frac{\tilde{y}x' - y'\tilde{x}}{x' - \tilde{x}} + \frac{y' - \tilde{y}}{x' - \tilde{x}} x \text{ according as} \\ x \begin{cases} \neq \\ = \end{cases} x' = \bar{x} = \underline{x} \text{ for } \underline{y} \leq y' \leq \bar{y},$$

which is the precise statement of the desired conclusion, shown in Figure A1.

The second additional property -- that the indifference curve passing through  $(\underline{x}, \underline{y})$  lies strictly below  $(x^*, y^*)$  -- was already guaranteed by the value restrictions on  $\underline{f}$ , specifically, that  $\underline{f}(\underline{x}) = \underline{y}$  and  $\underline{f}(x^*) < y^*$ , also shown in Figure A1.

There are two additional points related to the foregoing construction which merit at least passing comment: (i) The same basic technique can be employed to justify the offer curve originating at  $(\tilde{x}, \tilde{y})$  having a nonvertical linear segment, say,

$$y = a + bx \text{ for } 0 < x^1 \leq x \leq x^2 < \tilde{x},$$

provided, for instance, that  $b < \frac{y^2 - \tilde{y}}{x^2 - \tilde{x}} < 0$ . Since the details of such a construction are not especially material to any of our present objectives, we will not elaborate them here.

(ii) The utility function implicitly defined by (A7) is not differentiable when either  $(x, y) = (x, \underline{f}(x)) \geq 0$  or  $(x, y) = (x, \bar{f}(x)) \geq 0$ .



Wayne Shafer has suggested an alternative procedure for defining a utility function which exhibits all the same properties, but which is also both differentiable and homothetic. The essence of Wayne's clever idea is displayed in Figure 2A, and in bare outline (i.e., without any subtleties) goes as follows: Pick a function  $f : \{x : x \geq \underline{x}\} \rightarrow \mathbb{R}$  which yields  $f(\underline{x}) = \underline{y}$  and  $f'(\underline{x}) = \frac{\underline{y} - \tilde{y}}{\underline{x} - \tilde{x}}$ . Next, extend this function leftward from  $x = \underline{x}$  by solving the ordinary differential equation

$$\frac{dy}{dx} = \frac{(y/x)\underline{x} - \tilde{y}}{\underline{x} - \tilde{x}} \text{ for } x \leq \underline{x}$$

with initial condition  $y(\underline{x}) = \underline{y}$ ; the solution is (no surprise) basically a power function

$$y = f(x) = \frac{\tilde{y}}{\tilde{x}} x + \left( \frac{\underline{y}\tilde{x} - \tilde{y}\underline{x}}{\tilde{x}} \right) \left( \frac{x}{\underline{x}} \right)^{\frac{\underline{x}}{\underline{x} - \tilde{x}}} \text{ for } x \leq \underline{x}.$$

Finally, define the indifference curves as appropriate radial projections of  $(x, f(x))$  from 0, and label them in some smooth, monotonic fashion, for instance, according to the y-coordinate of their intersection with the ray  $y = (y/x)x$ .

Our particular algorithm has two advantages over Wayne's, the first minor, the second not so minor. In the first place, it conveniently enables satisfying the additional dominance condition  $U(x^*, y^*) > U(\underline{x}, \underline{y})$  (or, more generally, satisfying various other additional restrictions on the indifference map).

In the second place, and more importantly, it can be used virtually unaltered to construct the sort of indifference map underlying the nonconvexity example described in Figure 11. In fact,

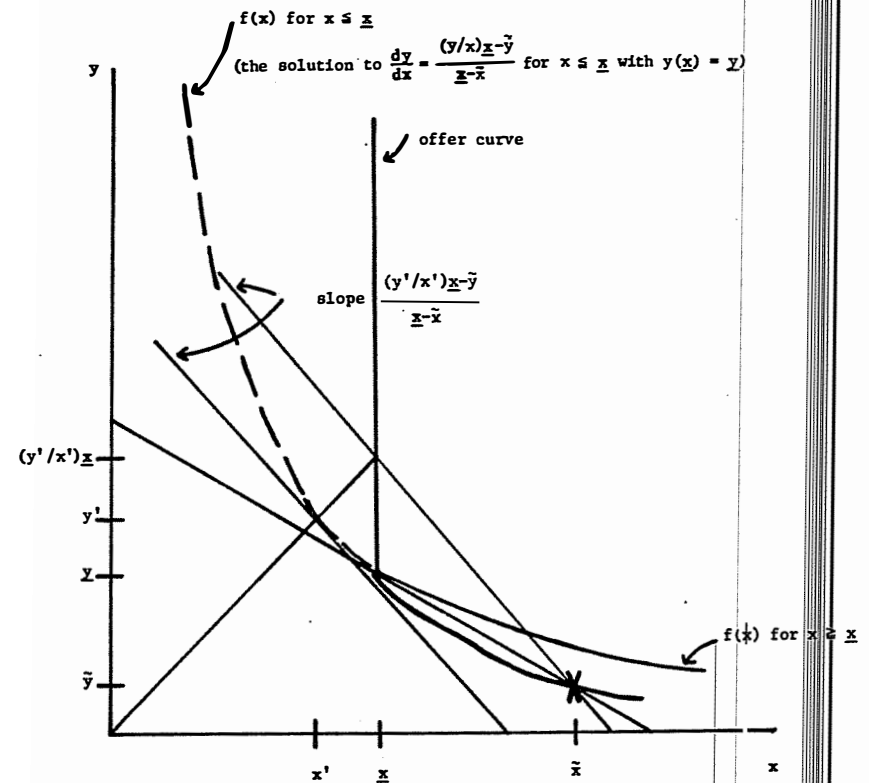


Figure A2. Construction of a Utility Function which Yields an Essentially Vertical Offer Curve and is both Differentiable and Homothetic.

the only significant change required is in the choice of  $\bar{f}$ : Given  $(\hat{x}, \hat{y}) > 0$  such that  $\hat{x} < \bar{x}$  and  $\frac{\hat{y}-\bar{y}}{\hat{x}-\bar{x}} = \frac{\bar{y}-\bar{y}}{\bar{x}-\bar{x}}$ ,  $\bar{f}$  now needs simply be specified, for instance, to be differentiable and to satisfy

$$(A10) \quad \bar{f}(x) \begin{cases} > \\ = \end{cases} \frac{\bar{y}\bar{x}-\bar{y}\hat{x}}{\bar{x}-\hat{x}} + \frac{\bar{y}-\bar{y}}{\bar{x}-\bar{x}} x \text{ according as } x \begin{cases} \neq \\ = \end{cases} \bar{x} \text{ or } \hat{x},$$

as shown in Figure A3. (Compare with the second inequality in (A2), and its representation in Figure A1.) Given such a specification, the earlier argument (where relevant) is identical down to the bottom line (A9), which only need be slightly modified to correspond with (A10)

$$(A9') \quad \dots x \begin{cases} \neq \\ = \end{cases} x' = \bar{x} = \underline{x} \quad (x \begin{cases} \neq \\ = \end{cases} \bar{x} \text{ or } \hat{x}) \text{ for } \underline{y} \leq y < \bar{y} \quad (y=\bar{y}).$$

In other words, the offer curve must now have the vertical segment with an upper endpoint discontinuity as depicted in Figure A3. (A separate argument, not spelled out here, establishes that the offer curve in the nonnegative region lying strictly above the prespecified curve  $y = \bar{f}(x)$  for  $x \geq 0$  must have the single-valuedness property shown as well, provided that  $\bar{f}$  is strictly convex on the interval  $[0, \hat{x}]$ .)

#### AIII. Truncated Offer Curves

It is worth briefly sketching an explicit procedure for obtaining the satiation example described in Figure 9; an obvious elaboration then yields that described in Figure 8. Thus, again employing the neutral notation  $(x, y)$ ,  $U$  and  $(\bar{x}, \bar{y})$ , we show here that it is possible to construct a well-behaved utility function which achieves a global maximum at a consumption vector  $(\bar{x}, \bar{y})$  such that

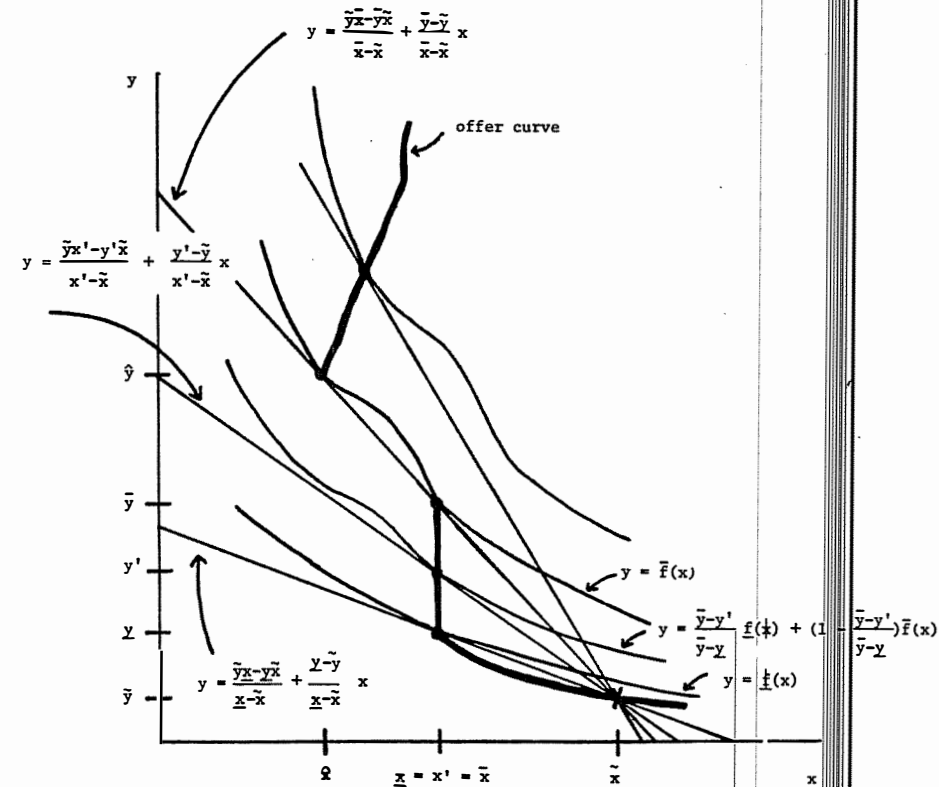


Figure A3. Construction of a Utility Function which Yields a Vertical Segment — with a Discontinuity at its Upper Endpoint — on the Offer Curve.

$\bar{x} < \tilde{x}$  and  $-1 < \frac{\bar{y}-\tilde{y}}{\bar{x}-\tilde{x}} < 0$ . Again, the basic nature of the construction is geometric, and can be completely captured in a diagram, as illustrated by Figure A4. An algorithm for this particular construction runs as follows: Pick an ellipse lying wholly within the positive quadrant, say,  $f(x,y) = 0$ . Then, let

$$x^\infty = \min \{x : f(x,y) = 0 \text{ for some } y > 0\}$$

and

$$y^0 = \min \{y : f(x,y) = 0 \text{ for some } x > 0\},$$

with corresponding abscissa  $y^\infty$  and ordinate  $x^0$ , respectively, and let  $(x^1, y^1)$  be such that  $x^\infty < x^1 < x^0$ ,  $y^0 < y^1 < y^\infty$ ,  $f(x^1, y^1) = 0$  and

$$\frac{\partial f(x^1, y^1)}{\partial x} / \frac{\partial f(x^1, y^1)}{\partial y} = -1.$$

(Refer to Figure A4; in this case the picture is surely much more informative than any accompanying algebra!) Next, pick any  $(\tilde{x}, \tilde{y})$  such that  $x^1 < \tilde{x} < x^0$ ,  $y^0 < \tilde{y} < y^1$  and  $f(\tilde{x}, \tilde{y}) = 0$ , so that

$$-1 < \frac{\partial f(\tilde{x}, \tilde{y})}{\partial x} / \frac{\partial f(\tilde{x}, \tilde{y})}{\partial y} < 0,$$

and -- given  $(\tilde{x}, \tilde{y})$  -- any  $(\bar{x}, \bar{y})$  lying strictly inside  $f(x,y) = 0$  but strictly below  $y = (\tilde{y}-\tilde{x}) - x$ , so that also

$$-1 < \frac{\bar{y}-\tilde{y}}{\bar{x}-\tilde{x}} < \frac{\partial f(\tilde{x}, \tilde{y})}{\partial x} / \frac{\partial f(\tilde{x}, \tilde{y})}{\partial y}.$$

Finally, once again define the indifference curves as radial projections, but here of the ellipse  $f(x,y) = 0$  from the point  $(\bar{x}, \bar{y})$ , and label them in some smooth, monotonic fashion, but here so that  $(\bar{x}, \bar{y})$  has

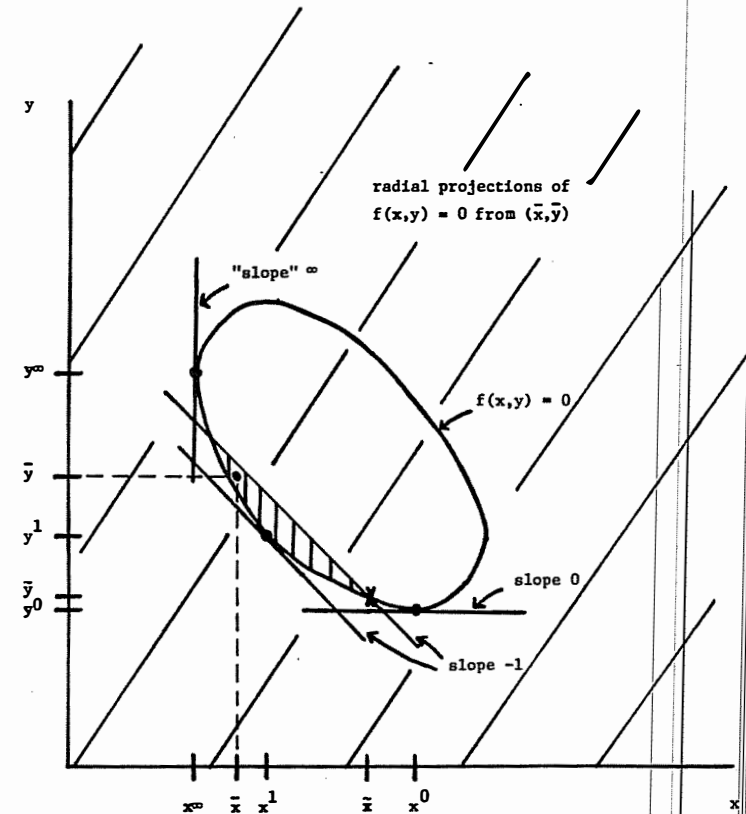


Figure A4. Construction of a Utility Function which Yields a Truncated Offer Curve.

the highest value, for instance, according to the lower y-coordinate of their intersection with the ray  $y = (\bar{y}/\bar{x})x$ .

Notice a couple of additional features of the utility function and endowment vector so constructed: (i) The corresponding offer curve will be backward bending (as drawn in Figures 8 and 9) if and only if

$$\left. \frac{\partial f(x,y)}{\partial x} / \frac{\partial f(x,y)}{\partial y} \right|_{(x,y) = (\bar{x}, \min\{y : f(\bar{x},y) = 0\})} > \frac{\bar{y}-\tilde{y}}{\bar{x}-\tilde{x}}.$$

(ii) The relationship between particular points on the corresponding offer curve and other strategically chosen points on the indifference map (as drawn, for instance, in Figure 8) can -- within limits -- be determined by suitably varying  $f$ ,  $(\tilde{x},\tilde{y})$  and  $(\bar{x},\bar{y})$ .